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Local polynomial convexity of graphs of functions in several variables

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ABSTRACT

In this paper, we investigate the locally polynomial convexity of graphs of smooth functions in several variables. We also give a sufficient condition for a real analytic function g defined near 0 in $\mathbb{C}$ which behaves like $\bar{z}^n$ near the origin so that the algebra generated by z^m and g is dense in the space of continuous functions on D for all disks D close enough to the origin in $\mathbb{C}$.

1. Introduction

We recall that for a given compact set K in $\mathbb{C}^n$, by $\hat{K}$ we denote the polynomial convex hull of K i.e.,

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K \text{ for every polynomial } p \text{ in } \mathbb{C}^n\}.$$

We say that K is polynomially convex if $\hat{K} = K$. A compact set K is called to be locally polynomially convex at $z \in K$ if there exists a closed ball $B(z)$ centered at z such that $B(z) \cap K$ is polynomially convex. The interest for studying polynomial convexity stems from the celebrated Oka–Weil approximation theorem (see [1], page 36) which states that holomorphic functions near a compact polynomially convex subset of $\mathbb{C}^n$ can be uniformly approximated by polynomials in $\mathbb{C}^n$. A compact set $K \subset \mathbb{C}$ is polynomially convex if $\mathbb{C} \setminus K$ is connected. In higher dimensions, there is no such topological characterization of polynomially convex sets, and it is usual difficult to determine whether a given compact subset is polynomially convex. By a well-known result of Wermer ([12]; see also [1], Theorem. 17.1), every totally real manifold is locally polynomially convex. Recall that a $\mathcal{C}^1$ smooth real manifold M is called totally real at $p \in M$ if the real tangent space T_pM contains no complex line. In this paper, we are concerned with local polynomial convexity at the origin of the graph Γ_f of a $\mathcal{C}^2$ smooth function f near $0 \in \mathbb{C}^n$ such that $f(0) = 0$. By the theorem

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of Wermer just cited, we know that if $\dfrac{\partial f}{\partial \overline{z}_i}(0) \neq 0$ for all $i = 1, 2, \ldots, n$ then Γ_f is locally polynomially convex at the origin of $\mathbb{C}^{n+1}$. Thus it remains to consider the case where $\dfrac{\partial f}{\partial \overline{z}_i}(0) = 0$ for some i. Our study is motivated by a similar problem in one complex variable. More precisely, let f be a $\mathcal{C}^2$ smooth function near $0 \in \mathbb{C}$ such that $f(0) = 0$. Under certain condition of f, one can show that Γ_f is locally polynomially convex at the origin of $\mathbb{C}^2$. The work associated with these directions of research is too numerous to list here; instead, the reader is referred to [2,3,13] and the references given therein.

In Section 3, we will refine the technique from [5] to attack the problem in several variables. For the readers convenience, we repeat a reasoning due to [5]. Firstly, we construct nonnegative smooth functions vanishing exactly on Γ_f. These functions are, in general, plurisubharmonic only on open sets whose boundaries contain the origin. Secondly, under some technical assumptions, we may add small strictly plurisubharmonic functions to obtain plurisubharmonic functions on certain open sets containing the (local) polynomially convex hull of Γ_f. Finally, by invoking the nontrivial fact about equivalence of plurisubharmonic hull and polynomial hulls, we can conclude that Γ_f is locally polynomially convex at the origin. In this vein, we obtain some known results in one variable. We also give some examples to show that our results are effective.

In Section 4, we shall present some results about locally uniform approximation of continuous functions. Let D be a small closed disk in the complex plane, centered at the origin and g be a $\mathcal{C}^2$ function on D which behaves like $\overline{z}^n$ near the origin. By $[z^m, g; D]$ we denote the function algebra consisting of uniform limits on D of all polynomials in z^m and g. Our goal is to find conditions on g to ensure that $[z^m, g; D] = C(D)$, where $C(D)$ is the set of continuous complex valued functions on D. Of course a necessary condition is that the two functions z^m and g must separate points of D. However, this condition is far from sufficient. Indeed, it can be shown that $[z^m, g; D] \neq C(D)$ for some choices of g (see [9,10], etc.), while for other choices of g we have $[z^m, g; D] = C(D)$ ([4,6], etc.). Our goal is to give generalized class of functions g defined near the origin in $\mathbb{C}$ such that $[z^m, g; D] = C(D)$. As in the previous work, we rely heavily on the theory of polynomial convexity. It may be useful to recall the general scheme in proving $[z^m, g; D] = C(D)$ for appropriately chosen g. Roughly speaking we consider the compact set $\tilde{X}$ which is inverse of $X := \{(z^m, g) : z \in D\}$ under the proper polynomial mapping $(z, w) \mapsto (z^m, w)$. Then $\tilde{X}$ is a union of graphs (in $\mathbb{C}^2$) over D. If g behaves “nicely” near the origin then we could show that each graph is polynomially convex. Notice that we can not apply a well known result of Wermer as in [1] or [12], since each graph may fail to be totally real at the origin. Thus in this case the compact $\tilde{X}$ is a union of polynomially convex compact sets. If we could prove that $\tilde{X}$ is polynomially convex, then by some known results in [8] about approximation on totally real compact sets (possibly with singularities) we could show that every continuous function on $\tilde{X}$ can be approximated uniformly by polynomials. Since $\tilde{X}$ transforms nicely to X, by a well known lemma (see [10]) we may obtain the same conclusion on X. Hence, it remains to decide the polynomial convexity of $\tilde{X}$. For this, we shall use an appropriate tool which is the version of Kallin lemma.

2. Some technical lemmas

Consider the function $g : [0, +\infty) \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} e^{\frac{-1}{t}}, & t > 0 \\ 0 & t = 0. \end{cases}$$

For each $x \geq 0$ set

$$\chi(x) = \int_0^x g(t)dt.$$