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Abel means for orthogonal expansions of distributions on spheres, balls and simplices

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ABSTRACT

A point is out of the support of a distribution on the open standard simplex if and only if the orthogonal expansion of the distribution is uniformly summable in Abel means to zero on a neighborhood of the point. A similar equivalence holds for Fourier–Laplace series on the unit sphere and orthogonal expansions on the open unit ball.

1. Introduction

In [10] Walter proved that the Fourier series

$$\sum_{-N\leq k\leq N} \mathcal{F}T(k)\mathrm{e}^{2\pi ixk}$$

of a distribution T on the circle $\mathbf{S}^1$ is Cesàro-summable to zero for all x out of the support of T. However, this is not sufficient to characterize the support of T, since, as Walter himself remarks, the Fourier series of the first derivative of the Dirac measure at a point $s \in \mathbf{S}^1$, δ_s', is summable in Cesàro means of order 2 to zero everywhere on $\mathbf{S}^1$. A way to completely characterize the support of T is to use asymmetric partial sums:

$$\sum_{-N\leq k\leq aN} \mathcal{F}T(k)\mathrm{e}^{2\pi ixk}, \quad \forall a > 0,$$

as shown by Vindas and Estrada in [8] and [9].

Using only symmetric partial sums, we see that a point x is out of the support of T if and only if the Fourier series of T is *uniformly* Cesàro-summable to zero on a neighborhood of x. We established this for

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the general case of a distribution T on $\mathbf{S}^{n-1}$ ($n \geq 2$) and its Fourier–Laplace series and obtained as corollary a similar equivalence for orthogonal expansions on the ball [4].

Recently, Dai, Wang and Ye, building on previous works of Xu (e.g. [12]), have shown how results about orthogonal expansions on the simplex can be obtained from similar results about orthogonal expansions on the ball [2].

Here, we will show that a point x is out of the support of a distribution T on the open unit simplex if and only if the orthogonal expansion of T is uniformly summable in Abel means to zero on a neighborhood of x. For that, we first prove a similar result about the Fourier–Laplace series of a distribution on the sphere (this was given as a remark without proof in [4, p. 764]). From this equivalence we will derive one about orthogonal expansions on the ball and finally, following [2], the desired one on the simplex.

The paper is divided as follows. Before proving the result on the sphere, the ball and the simplex in Sections 3, 5 and 7 respectively, we give the necessary definitions and tools in Sections 2, 4 and 6 respectively.

2. Preliminaries for the sphere

We write $\mathbf{S}^{n-1}$ for the unit sphere in $\mathbf{R}^n$ ($n \geq 2$) and $d\sigma_{n-1}$ for the measure on $\mathbf{S}^{n-1}$ induced by the Lebesgue measure on $\mathbf{R}^n$, so that

$$\omega_{n-1} := \int_{\mathbf{S}^{n-1}} d\sigma_{n-1}(\eta) = 2\pi^{n/2}/\Gamma(n/2).$$

We define a distance d on $\mathbf{S}^{n-1}$ by $d(\zeta, \eta) := 1 - (\zeta|\eta)$, where $(\,.\,|\,.\,)$ is the euclidean scalar product in $\mathbf{R}^n$; we put $\|x\| := (x|x)$. A spherical harmonic of degree ℓ on $\mathbf{S}^{n-1}$ ($\ell \in \mathbf{N}_0$) is the restriction to $\mathbf{S}^{n-1}$ of a polynomial on $\mathbf{R}^n$ which is harmonic and homogeneous of degree ℓ. We write $\mathcal{V}_\ell(\mathbf{S}^{n-1})$ for the vector space of spherical harmonics of degree ℓ; its dimension is

$$v_\ell^n := \dim_{\mathbf{C}} \mathcal{V}_\ell(\mathbf{S}^{n-1}) = \frac{(2\ell + n - 2)(n + \ell - 3)!}{(n-2)!\, l!} = \frac{2\ell^{n-2}}{(n-2)!} + O(\ell^{n-3}).$$

Two spherical harmonics of different degrees are orthogonal with respect to the scalar product $(\,.\,|\,.\,)_2$ of $L^2(\mathbf{S}^{n-1}, d\sigma_{n-1})$. Let $(E_1^\ell, \ldots, E_{v_\ell}^\ell)$ be an orthonormal basis of $\mathcal{V}_\ell(\mathbf{S}^{n-1})$. If $f \in L^2(\mathbf{S}^{n-1})$, the series

$$\sum_{\ell=0}^{+\infty} \sum_{j=1}^{v_\ell} (f|E_j^\ell)_2 \cdot E_j^\ell$$

is the orthogonal expansion of f (also called Fourier–Laplace series of f); it converges to f in square mean; and

$$\mathrm{proj}_\ell^{\mathbf{S}}(f) := \sum_{j=1}^{v_\ell} (f|E_j^\ell)_2 \cdot E_j^\ell$$

is the orthogonal projection of f on $\mathcal{V}_\ell(\mathbf{S}^{n-1})$. For $\zeta \in \mathbf{S}^{n-1}$,

$$\mathrm{proj}_\ell^{\mathbf{S}}(f)(\zeta) = \int_{\mathbf{S}^{n-1}} Z_\ell(\zeta, \eta) f(\eta) d\sigma_{n-1}(\eta),$$

where $Z_\ell(\zeta, \eta) := \sum_{j=1}^{v_\ell} E_j^\ell(\zeta)\overline{E_j^\ell(\eta)}$ is the *zonal with pole ζ of degree ℓ*.

If f is a function defined on $\mathbf{S}^{n-1}$, we write $f\!\uparrow$ for the homogeneous function of degree 0 defined on $\mathbf{R}^n \setminus \{0\}$ by $(f\!\uparrow)(x) := f(x/\|x\|)$. Conversely, if g is a function defined on $\mathbf{R}^n \setminus \{0\}$ we note $g\!\downarrow$ its restriction