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Generalized Newtonian fluid flow through a porous medium

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ABSTRACT

We present a model for generalized Newtonian fluid flow through a porous medium. In the model the dependence of the fluid viscosity on the velocity is replaced by a dependence on a smoothed (locally averaged) velocity. With appropriate assumptions on the smoothed velocity, existence of a solution to the model is shown. Two examples of smoothing operators are presented in the appendices. A numerical approximation scheme is presented and an a priori error estimate derived. A numerical example is given illustrating the approximation scheme and the a priori error estimate.

1. Introduction

Of interest in this article is the modeling and approximation of generalized Newtonian fluid flow through a porous medium. Darcy's modeling equations for a steady-state fluid flow through a porous medium, Ω, are

$$\nu_{eff}\, K^{-1}\,\mathbf{u} + \nabla p = 0, \text{ in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \text{ in } \Omega, \tag{1.2}$$

where $\mathbf{u}$ and p denote the velocity and pressure of the fluid, respectively. $K(\mathbf{x})$ in (1.1) represents the permeability of the medium at $\mathbf{x} \in \Omega$, which is assumed to be a symmetric, positive definite tensor. As our investigations are not concerned with K, we assume that K is of the form $k(\mathbf{x})\mathbf{I}$ where $k(\mathbf{x})$ is a Lipschitz continuous, positive, bounded and bounded away from zero, scalar function. ν_{eff} in (1.1) represents the effective viscosity of the fluid.

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In the case of a Newtonian fluid in a porous medium we have that ν_{eff} is a positive constant. For a generalized Newtonian fluid various models for ν_{eff} have been proposed in the literature. (See for example, [14,16] and the references cited therein.) Based upon dimensional analysis most models assume that ν_{eff} is a function of $|\mathbf{u}|/(\sqrt{|K|}m_c)$, where m_c is a constant related to the internal structure of the porous media. Models for ν_{eff} include [14,16]:

$$\text{Power Law Model:} \quad \nu_{eff}(|\mathbf{u}|) = c_\nu |\mathbf{u}|^{r-2}, \qquad \text{Cross Model:} \quad \nu_{eff}(|\mathbf{u}|) = \nu_\infty + \frac{\nu_0 - \nu_\infty}{1 + c_\nu |\mathbf{u}|^{2-r}}, \tag{1.3}$$

where c_ν, ν_0, ν_∞ and r are fluid dependent constants. For shear thinning fluids $1 < r < 2$. (In modeling the viscosity of shear thinning fluids the Power Law model suffers the criticism that as $|\mathbf{u}| \to 0$ $\nu_{eff} \to \infty$.)

For the case of a Newtonian fluid (1.1), (1.2) are well studied. The two standard approaches in analyzing (1.1), (1.2) are: (i) study (1.1), (1.2) as a mixed formulation problem for $\mathbf{u}$ and p (either $(\mathbf{u}, p) \in H_{div}(\Omega) \times L^2(\Omega)$, or $(\mathbf{u}, p) \in L^2(\Omega) \times H^1(\Omega)$), or (ii) use (1.2) to eliminate $\mathbf{u}$ in (1.1) to obtain a generalized Laplace's equation for p.

For generalized Newtonian fluids, with $\nu_{eff} = \nu_{eff}(|\mathbf{u}|)$, assumptions are required on ν_{eff} in order to establish existence and uniqueness of solutions. Typical assumptions are uniform continuity of $\nu_{eff}(|\mathbf{u}|)\mathbf{u}$ and strong monotonicity of $\nu_{eff}(|\mathbf{u}|)$ [8,9,11], i.e., there exists $C > 0$ such that

$$|\nu_{eff}(|\mathbf{u}|)\mathbf{u} - \nu_{eff}(|\mathbf{v}|)\mathbf{v}| \leq C|\mathbf{u} - \mathbf{v}|, \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \tag{1.4}$$

$$(\nu_{eff}(|\mathbf{u}|)\mathbf{u} - \nu_{eff}(|\mathbf{v}|)\mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \geq C(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}), \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d. \tag{1.5}$$

A more general setting where the fluid rheology is defined implicitly has been analyzed in [6,7]. The case where the fluid viscosity depends on the shear rate and pressure has been studied in [15,13]. For both of these cases additional structure beyond (1.4) and (1.5) is required in order to establish existence and uniqueness of a solution.

A nonlinear Darcy fluid flow problem, with a permeability dependent upon the pressure was investigated by Azaïez, Ben Belgacem, Bernardi, and Chorfi [2], and Girault, Murat, and Salgado [12]. For a Lipschitz continuous permeability function, bounded above and bounded away from zero, existence of a solution $(\mathbf{u}, p) \in L^2(\Omega) \times H^1(\Omega)$ was established. Important in handling the nonlinear permeability function, in establishing existence of a solution, was the property that $p \in H^1(\Omega)$. In [2] the authors also investigated a spectral numerical approximation scheme for the nonlinear Darcy problem, assuming an axisymmetric domain Ω. A convergence analysis for the finite element discretization of that problem was given in [12].

Our interest in this paper is in relaxing the assumptions (1.4) and (1.5). Specifically, our interest is assuming that $\nu_{eff}(\cdot)$ is only Lipschitz continuous and both bounded above and bounded away from zero. However, relaxing the conditions (1.4) and (1.5) requires us to make an additional assumption regarding the argument of $\nu_{eff}(\cdot)$. In order to obtain a modeling system of equations for which a solution can be shown to exist, we replace $\mathbf{u}$ in $\nu_{eff}(|\mathbf{u}|)$ by a *smoothed* velocity, $\mathbf{u}^s$. The approach of regularizing the model with the introduction of $\mathbf{u}^s$ is, in part, motivated by the fact that the Darcy fluid flow equations can be derived by *averaging*, e.g. volume averaging [19], homogenization [1], or mixture theory [17].

Presented in Appendices A and B are two smoothing operators for $\mathbf{u}$. One is a local averaging operator, whereby $\mathbf{u}^s(\mathbf{x})$ is obtained by averaging $\mathbf{u}$ in a neighborhood of $\mathbf{x}$. The second smoothing operator, which is nonlocal, computes $\mathbf{u}^s(\mathbf{x})$ using a differential filter applied to $\mathbf{u}$. That is, $\mathbf{u}^s$ is given by the solution to an elliptic differential equation whose right hand side is $\mathbf{u}$. For establishing the existence of a solution to (1.1)–(1.2), the key property of the smoothing operators is that they transform a weakly convergent sequence in $L^2(\Omega)$ into a sequence which converges strongly in $L^\infty(\Omega)$.