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Caffarelli–Kohn–Nirenberg type equations of fourth order with the critical exponent and Rellich potential

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ABSTRACT

We study the existence/nonexistence of positive solutions of

$$\Delta^2 u - \mu \frac{u}{|x|^4} = \frac{|u|^{q_\beta - 2}u}{|x|^\beta} \quad \text{in } \Omega,$$

where Ω is a bounded domain and $N \geq 5$, $q_\beta = \frac{2(N-\beta)}{N-4}$, $0 \leq \beta < 4$ and $0 \leq \mu < \left(\frac{N(N-4)}{4}\right)^2$. We prove the nonexistence result when Ω is an open subset of $\mathbb{R}^N$, which is star-shaped with respect to the origin. We also study the existence of positive solutions when Ω is a smooth bounded domain with a nontrivial topology and $\beta = 0$, $\mu \in (0, \mu_0)$, for certain $\mu_0 < \left(\frac{N(N-4)}{4}\right)^2$ and $N \geq 8$. Different behaviors are obtained for Palais–Smale sequences depending on whether $\beta = 0$ or $\beta > 0$.

1. Introduction

In this study, we consider the singular semilinear fourth order elliptic problem:

$$\begin{cases} \Delta^2 u - \mu \frac{u}{|x|^4} = \frac{|u|^{q_\beta - 2}u}{|x|^\beta} & \text{in } \Omega, \\ u \in H_0^2(\Omega), \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Delta^2 u = \Delta(\Delta u)$, Ω is a smooth bounded domain and

$$N \geq 5, \qquad q_\beta = \frac{2(N-\beta)}{N-4}, \quad 0 \leq \beta < 4 \quad \text{and} \quad 0 \leq \mu < \bar{\mu} = \left(\frac{N(N-4)}{4}\right)^2. \tag{1.2}$$

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Semilinear elliptic equations with a biharmonic operator arise in continuum mechanics, biophysics, and differential geometry, particularly in the modeling of thin elastic plates and clamped plates, as well as in the study of the Paneitz–Branson equation and the Willmore equation (see [13] and the references therein for more details).

Definition 1.1. We say that $u \in H_0^2(\Omega)$ is a solution to (1.1) if $u > 0$ in Ω and satisfies

$$\int_\Omega \left[\Delta u \Delta \phi - \mu \frac{u\phi}{|x|^4}\right] dx = \int_\Omega \frac{|u|^{q_\beta - 2} u\phi}{|x|^\beta} dx \quad \forall \phi \in H_0^2(\Omega).$$

Equivalently, u is a critical point of the functional

$$I_\mu(u) = \frac{1}{2}\int_\Omega \left[|\Delta u|^2 - \mu \frac{|u|^2}{|x|^4}\right] dx - \frac{1}{q_\beta}\int_\Omega \frac{|u|^{q_\beta}}{|x|^\beta} dx \quad u \in H_0^2(\Omega). \tag{1.3}$$

I_μ is a well defined C^1 functional in $H_0^2(\Omega)$ due to the following Rellich inequality [19,18]:

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \bar{\mu} \int_{\mathbb{R}^N} |x|^{-4} |u|^2 dx \quad \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{1.4}$$

and the Caffarelli–Kohn–Nirenberg (CKN) inequality of fourth order [3,6,8]:

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq C \left(\int_{\mathbb{R}^N} |x|^{-\beta} |u|^{q_\beta} dx\right)^{2/q_\beta} \qquad \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{1.5}$$

where $C = C(N, \beta) > 0$. The Rellich inequality is a generalization of the following Hardy inequality:

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{1.6}$$

Remark: It is well known that when Ω is a smooth bounded domain, the Hardy inequality holds for every $u \in H_0^1(\Omega)$, but the best constant $\left(\frac{N-2}{2}\right)^2$ is never achieved.

From previous studies, we know that the usual norm in $H^2(\Omega)$ is $\left(\int_\Omega \sum_{0 \leq |\alpha| \leq 2} |D^\alpha u|^2 dx\right)^{\frac{1}{2}}$. From interpolation theory, we can neglect the intermediate derivates and we find that

$$\|u\|_{H^2(\Omega)} = \left(\int_\Omega |u|^2 dx + \int_\Omega |D^2 u|^2 dx\right)^{\frac{1}{2}} \tag{1.7}$$

defines a norm that is equivalent to the usual norm in $H^2(\Omega)$ (see [1]). As Ω is a smooth bounded domain and $H_0^2(\Omega)$ is the closure of $C_0^\infty(\Omega)$ w.r.t. the norm in $H^2(\Omega)$, then by invoking [13, Theorem 2.2] we find that