J. Math. Anal. Appl. 433 (2016) 701–705

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic analysis of the iterative power means

Tomislav Burić

University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, Croatia

A R T I C L E   I N F O

Article history:
Received 18 March 2015
Available online 11 August 2015
Submitted by M. Laczkovich

Keywords:
Asymptotic expansion
Power mean
Compound mean
Arithmetic–geometric mean

A B S T R A C T

We investigate the asymptotic expansion of the compound mean obtained by the iterative process of two power means. We present the stationary and convergence properties of the coefficients in the expansions.

1. Introduction and motivation

The arithmetic–geometric mean is a well-known interesting mean which is defined by the iterative process of the arithmetic and geometric means of two positive numbers s and t in a following way. Let $a_0 = t$, $g_0 = s$ and

$$a_n = \frac{a_{n-1} + g_{n-1}}{2}, \quad g_n = \sqrt{a_{n-1}\, g_{n-1}}, \quad n \geq 1. \tag{1.1}$$

Then the increasing sequence g_n and the decreasing sequence a_n both converge to the same limit which is called the arithmetic–geometric mean.

This iterative process can be generalized to any two bivariate means M and N. Let us define $M_0(s,t) = s$, $N_0(s,t) = t$ and

$$M_n(s,t) = M(M_{n-1}, N_{n-1}), \quad N_n(s,t) = N(M_{n-1}, N_{n-1}), \quad n \geq 1. \tag{1.2}$$

If both limits of these sequences exist and are equal, then that common value is called the compound mean of s and t and is denoted by $M \otimes N(t,s)$. Details about this concept and existence of compound means can be found in [2, Ch. VI.3].

E-mail address: tomislav.buric@fer.hr.

http://dx.doi.org/10.1016/j.jmaa.2015.08.020

In a recent paper [4], the authors derived asymptotic expansion of the arithmetic–geometric mean $A \otimes G$, and analyzed convergence and stationary properties of the coefficients in this expansion. Later, in [3], the same method is used to analyze the iterative process of the arithmetic-harmonic and geometric-harmonic means which also showed the same interesting stationary properties of the coefficients in their asymptotic expansions.

Recall that the arithmetic, geometric and harmonic means are special cases of the more general r-th power mean which is defined for all real $s, t \geq 0$ by

$$\mathcal{M}_r(s,t) = \begin{cases} \left(\dfrac{s^r + t^r}{2}\right)^{\frac{1}{r}} & , \ r \neq 0, \\ \sqrt{st} & , \ r = 0. \end{cases} \tag{1.3}$$

Therefore, $\mathcal{M}_1$ equals the arithmetic mean, $\mathcal{M}_{-1}$ the harmonic mean, and clearly, the geometric mean $G = \mathcal{M}_0$ is obtained as limit of means $\mathcal{M}_r$ when $r \to 0$.

In [1], the authors gave a complete asymptotic expansion of $\mathcal{M}_r$ with coefficients expressed in terms of partial and complete Bell polynomials. Using the new techniques, this expansion is also derived in [9] and the effective recursive formula for calculating coefficients in this expansion was presented.

The main aim of this paper is to analyze asymptotic behaviour of the iterative process of two general power means. We shall derive coefficients in the asymptotic expansion of the compound mean $\mathcal{M}_p \otimes \mathcal{M}_r$ and show that the stationary properties proved for the compounds of arithmetic, geometric and harmonic means also hold in the general case of any two power means. Moreover, we shall prove faster convergence properties than proved in papers [3,4].

2. Asymptotic expansion of iterative power means

Let $P = \mathcal{M}_p$ and $Q = \mathcal{M}_q$ be any two power means. To obtain the compound mean $P \otimes G$, we shall follow its iterative process in the following notion:

$$P_n = P(P_{n-1}, Q_{n-1}), \quad Q_n = Q(P_{n-1}, Q_{n-1}). \tag{2.1}$$

Let means in the n-th iteration have the following asymptotic expansions:

$$P_n(x+s, x+t) = x \sum_{k=0}^{\infty} a_k^{(n)}(s,t) x^{-k}, \quad x \to \infty, \tag{2.2}$$

$$Q_n(x+s, x+t) = x \sum_{k=0}^{\infty} b_k^{(n)}(s,t) x^{-k}, \quad x \to \infty, \tag{2.3}$$

where $a_k^{(n)}$ and $b_k^{(n)}$ are homogeneous polynomials of degree k of the variables s and t.

We shall prove that the sequences $a_k^{(n)}$ and $b_k^{(n)}$ converge to the same limit $c_k(t,s)$ when $n \to \infty$, i.e. they converge to the coefficients in the asymptotic expansion of the compound mean:

$$P \otimes Q(x+s, x+t) = x \sum_{k=0}^{\infty} c_k(s,t) x^{-k}, \quad x \to \infty. \tag{2.4}$$

This convergence of the coefficients is very fast, in fact, the following theorem holds true.