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Asymptotically periodic superquadratic Hamiltonian systems ☆

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A R T I C L E I N F O

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A B S T R A C T

We obtain existence of solutions for the system

$$\begin{cases} -\Delta u + V(x)u = F_v(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = F_u(x,u,v), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $V \in C(\mathbb{R}^N, (0,\infty))$ is periodic and $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ is superquadratic at infinity. We consider the case that F is periodic and asymptotically periodic. In the proofs we apply variational techniques.

1. Introduction

In this paper we prove existence of solutions for the following Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = F_v(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = F_u(x,u,v), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and F_u, F_v stand for the partial derivatives of F with respect to the second and third variable.

Recently, many authors have used variational techniques to study the system (1.1) and its variants. For the bounded domain case we refer the reader to [4,6,7,11,12] and references therein, while for the whole space $\mathbb{R}^N$ we quote the papers [2,5,8,14,18,22], among others. In these works, a huge machinery is needed to obtain existence and multiplicity of solutions: fractional Sobolev spaces, reduction methods, the generalized Mountain Pass Theorem, radial approaches and many others.

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One of the main difficulties in dealing with (1.1) relies on the lack of compactness due to the unboundedness of the domain. In some of the above quoted papers this difficulty was overcome by imposing periodicity both on the potential V and on the nonlinearity F. Here, we consider not only the periodic case for F, but also the asymptotically periodic case. The main assumption on $V \in C(\mathbb{R}^N, \mathbb{R})$ is the following

(V_0) $V(x) = V(x_1, \ldots, x_N)$ is positive and 1-periodic in the variables $x_1, \ldots, x_N$.

In our first result, we consider the periodic problem. We start by assuming the following growth conditions on the nonlinearity F:

(F_0) $F(x, z)$ is 1-periodic in the variables $x_1, x_2, \ldots, x_N$;
(F_1) there exist $c > 0$ and $p \in (2, 2N/(N-2))$ such that

$$|F_z(x, z)| \leq c(1 + |z|^{p-1}), \quad \text{for each } (x, z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $F_z(x, z) = (F_u(x, z), F_v(x, z)) \in \mathbb{R}^2$;
(F_2) $F_z(x, z) = o(|z|)$ as $|z| \to 0$, uniformly for $x \in \mathbb{R}^N$.

We are interested in the case that F is superquadratic at infinity. Since the seminal work of Ambrosetti and Rabinowitz [1], superquadratic problems (at infinity) are subject to intensive studies. The superlinear condition introduced in [1] reads as

(AR) there exists $\mu > 2$ such that, for each $x \in \mathbb{R}^N$, $z \in \mathbb{R}^2 \setminus \{0\}$, there holds

$$0 < \mu F(x, z) \leq F_z(x, z) \cdot z,$$

where the dot stands for the scalar product in $\mathbb{R}^2$. It is well known that is condition provides boundedness for the Palais–Smale sequences of the energy functional. A straightforward calculation shows that (AR) implies $F(x, z) \geq c|z|^\mu$ for large values of $|z|$, and therefore it is natural to consider the weaker assumption

(F_3) $\frac{F(x,z)}{|z|^2} \to \infty$ as $|z| \to \infty$, uniformly for $x \in \mathbb{R}^N$.

There are many results concerning the scalar version of (1.1) and considering the above superquadratic growth condition instead of (AR). Most of them use the Nehari approach for which the authors assume a monotonicity condition for $f(x, t)/t$. In our first result, we follow an analogous approach and therefore we need a version of this monotonicity condition for the nonlinearity F. We assume that

(F_4) there exists $g : \mathbb{R}^N \times \mathbb{R}^+ \to [0, +\infty)$ increasing in the second variable such that

$$F_z(x, z) = g(x, |z|)z, \quad \text{for each } (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

We recall that, for some suitable Banach space E (see Section 3), the weak solutions of problem (1.1) are critical points of the C^1-functional

$$I(z) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\, dx - \int_{\mathbb{R}^N} F(x, z) dx,$$