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Walsh and wavelet methods for differential equations on the Cantor group

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ABSTRACT

Ordinary and partial differential equations for unknown functions defined on the Cantor dyadic group are studied. We consider two types of equations: related to the Gibbs derivatives and to the fractional pseudo differential operators. Since the Cantor group is an ultrametric space, pseudo differential operators have spacial properties and are of interest for some applications to models of complex systems, e.g., ultrametric diffusion models in biophysics. We find solutions to the equations in classes of distributions and analyse under what assumptions these solutions are regular functions with some "good" properties. Haar wavelets are used to solve pseudo differential equation. It is very important that the Haar MRA coincides with the Shannon MRA on the Cantor group. To analyse solutions, specific computational method based on the multiresolution structure of the Haar basis was developed.

1. Introduction

Dyadic (Walsh) analysis is actively studied for the last three decades. The foundations of the Walsh theory are presented in monographs [10] and [16]. The theory may be stated in two equivalent forms: the functions under consideration are defined on the real semiline or they are defined on the Cantor group (see [10, §1.2]). We work in the latter form.

The dyadic wavelet theory is actively studied nowadays. In 1996 the concept of multiresolution analysis (MRA) for the Cantor group was introduced by Lang [14], who also developed a method for the con-

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struction MRA-based orthogonal wavelet bases. Later deep investigations of the foundation of the dyadic wavelet theory were added by Farkov and Protasov [7] (see also the recent papers [5,6] and the references therein).

A concept of dyadic derivative was introduced by Gibbs [8] in 1967. Numerous generalizations of this notion in different directions can be found in the literature, the surveys on this topic with extensive bibliographies were given in [19,18]. A notion of so-called modified Gibbs derivative we use in the present paper was introduced by Golubov in [9].

Dyadic analogues of the classical partial differential equations were considered by Butzer and Wagner [3,4]. They found a dyadic analogue of d'Alamber's solution to the one-dimensional homogeneous wave equation $\Phi^{[2]}_{x^2} = \Phi^{[2]}_{t^2}$, where $f^{[2]}$ denotes the Gibbs derivative of the second order. Note that both variables x and t are elements of the group G here. In the present paper, we consider PDE's, where the variable x is in G and the variable t, interpreted as time, is real. Equations of such kind in ultrametric spaces may be useful for some applications to models of complex systems, e.g., ultrametric diffusion models in biophysics (see, e.g., [13]), and the Cantor group is just an ultrametric space.

We study ordinary and partial differential equations with respect to both Gibbs and modified Gibbs derivatives (pseudo differential operators). In the case of Gibbs derivative, we consider some linear differential equations and the Cauchy problem for the homogeneous wave equation. The Walsh functions are eigenfunctions for the Gibbs differentiation operator. Due to this property, using Walsh expansions, we find all solutions of the equations in the class of periodic distributions. Also we investigate under what conditions the solution is a regular function with "good" properties (belonging to L_2 or continuity).

Unfortunately, the Gibbs derivative has the following drawback. Consider a simple equation, say $f^{[1]} = g$, where g is 1-periodic. To solve the solution one finds the Fourier–Walsh coefficients of $f^{[1]}$, which allows to restore f. However if g is restricted to the fundamental domain I, then one can repeat this trick using the Fourier–Walsh transform instead of the Fourier–Walsh coefficients, and then restore f. It appears that such a function f is not a compactly supported and it does not coincide on I with the periodic solution. The modified Gibbs differentiability is "more local" property of functions, an analog of the described drawback does not hold for the corresponding differential equations. Namely, if g is a 1-periodic function, f is an unknown function, then the periodic solution of the equation $\mathcal{D}f = g$ coincides on I with the solution of the equation $\mathcal{D}f = g\mathbb{1}_I$, where $\mathbb{1}_I$ is the characteristic function of I. Moreover, the solution of the later equation $\mathcal{D}f = g\mathbb{1}_I$ has the same support as the right-hand side. We consider fractional modified Gibbs derivatives (pseudo differential operators) $\mathcal{D}^\alpha$, $\alpha \in \mathbb{R}$. Since the Cantor group is an ultrametric space, pseudo differential operators have spacial properties and are of interest for some applications to models of complex systems, e.g., ultrametric diffusion models in biophysics (see, e.g., [15,2,13,12]).

We study ordinary and partial differential equations with respect to the fractional modified Gibbs derivative. In particular, we consider the Cauchy problem for the one-dimensional non-homogeneous heat equation. It appeared that all elements of the Haar basis are eigenfunctions for $\mathcal{D}^\alpha$. This allows to solve equations using Haar expansions. We find all solutions on a class of distributions and investigate under what conditions the solution is a regular function with "good" properties. The multiresolution structure of the Haar basis is essentially used in our technique. A specific computational method developed in Theorem 1 plays an important role.

The paper is organized as follows. First, we introduce necessary notations and basic facts of the Walsh analysis on the Cantor group. In Subsection 2.2 we discuss the Haar basis, its multiresolution structure, Haar and quasi-Haar expansions. In Section 3 we introduce classes of distributions on the Cantor group and establish some connections between them. It Section 4 we study differential equation with respect to the Gibbs derivative. It Section 5 we study differential equation with respect to the fractional modified Gibbs derivative.