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Asymptotic behavior results for solutions to some nonlinear difference equations

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ABSTRACT

We give asymptotic results for convergent solutions $\{x_n\}$ of (real or complex) difference equations $x_{n+1} = J\,x_n + f_n(x_n)$, where x_n is an m-vector, J is a constant $m \times m$ matrix and $f_n(y)$ is a vector valued function which is continuous in y for fixed n, and where $f_n(y)$ is small in a sense. In addition, we obtain asymptotic results for solutions $\{x_n\}$ of the Poincaré difference equation $x_{n+1} = (A + B_n)\,x_n$ where B_n satisfies $\|B_n\| = O(\eta^n)$ with $\eta \in (0, 1)$. An application illustrates the results.

1. Introduction

Consider the difference equation

$$x_{n+1} = J\,x_n + f_n(x_n), \tag{1}$$

where x_n is an m-vector, J is a constant $m \times m$ matrix and $f_n(y)$ is a vector valued function which is continuous in y for fixed n, and where $f_n(y)$ is small in some sense as $(n, \|y\|) \to (\infty, 0)$. Eq. (1) can be seen as a non-autonomous perturbation of the linear, constant coefficients equation $x_{n+1} = J\,x_n$. A natural question is, what can be said about the asymptotic behavior of solutions $\{x_n\}$ to (1) that converge to zero? Eq. (1) is very general and it includes as a particular case the (matrix) Poincaré equation

$$x_{n+1} = (A + B_n)\,x_n\,, \tag{2}$$

where x_n is an m-vector, and A, B_n are an $m \times m$ matrices for $n = 1, 2, \ldots$ such that $\|B_n\| \to 0$.

Eq. (1) has been studied by O. Perron, C.V. Coffman, and others [22,5], who obtained asymptotic behavior results for solutions $\{x_n\}$ of (1). Coffman's Theorem 5.1 in [5] is a refinement of results of Perron [22], and

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it states that if $\|f_n(y)\|/\|y\| \to 0$ as $(n, y) \to (\infty, 0)$, solutions $x = x_n$ to (1) that converge to zero either have $x_n = 0$ for large n or satisfy

$$\lim_{n\to\infty} |x_n|^{1/n} = |\lambda| \,,$$

where λ is an eigenvalue of J. Although not stated explicitly, Coffman further established (Theorems 5.1 and 8.1 of [5]) that $0 < |\lambda| < 1$ implies the existence of $\tilde{y} \in \mathbb{C}^d$ and $\kappa \in \mathbb{N}$ for which the following asymptotic relation holds:

$$x_n = J^n \, \tilde{y} + o(n^\kappa \, |\lambda|^n) \quad \text{as} \quad n \to \infty \,. \tag{3}$$

The origin of the study of relation (1) can be traced to Poincaré [27], who investigated the non-autonomous scalar linear difference equation

$$\zeta_{n+m} + p_{m-1,n} \, \zeta_{n+m-1} \; + \; \cdots \; + \; p_{1,n} \, \zeta_{n+1} \;\; = \;\; 0 \,, \tag{4}$$

for which the following limits are assumed to exist:

$$\lim_{n\to\infty} p_{m-1,n} = q_{m-1} \,, \qquad \ldots \qquad , \lim_{n\to\infty} p_{1,n} = q_1 \,. \tag{5}$$

Eq. (4) has a limiting equation

$$\zeta_{n+m} \; + \; q_{m-1} \, \zeta_{n+m-1} \; + \; \cdots \; + \; q_1 \, \zeta_{n+1} \;\; = \;\; 0. \tag{6}$$

Poincaré proved under the hypothesis that the roots of the characteristic equation of (6) have distinct moduli, that for every solution ζ_n of (4) for which $\zeta_n \neq 0$ for all large n, the ratios ζ_{n+1}/ζ_n approach one of the characteristic roots of (6) [27]. Poincaré assumed the coefficients $p_{\ell,n}$ in (4) to be rational functions of n, but this condition may be dropped if $p_{0,n}$ is required to be nonzero for all n (see the statement and proof of Theorem 2.13.1 in [1]). Poincaré's result was generalized by Perron [22], and by Gelfond et al. [14] to systems (1) and (2) (see [5]).

Since the mid 1990s there has been renewed interest in asymptotic results, see the book by Elaydi [7] and work cited therein, and see also [6,8–11], Pituk, Pituk et al., Bodine and Lutz, [23–26,21,2,3], Kalabusić–Kulenović [16], and many others [12,13,15,17–19].

In [2] R.P. Agarwal and M. Pituk studied the scalar equation (4) when the convergence in (5) is at a geometric rate. In [4], S. Bodine and D.A. Lutz greatly improved the estimates of Agarwal and Pituk, considering (matrix) Poincaré systems (2), again under the assumption of geometric convergence. The following result is Theorem 1 of [4] with changes in notation to match that of this paper.

Theorem BL1 *(Bodine–Lutz, 2009). Suppose that x_n is a solution of (2), where A is a constant and invertible matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$ repeated according to their multiplicity. Suppose that there exists $\eta \in (0,1)$ and $c > 0$ such that $\|B_n\| \leq c\,\eta^n$ for $n = 0, 1, \ldots$. For every fixed $i \in \{1, \ldots, d\}$ let $q_i := \max\{\left|\frac{\lambda_j}{\lambda_i}\right| : 1 \leq j \leq d$ such that $\left|\frac{\lambda_j}{\lambda_i}\right| < 1\}$ if at least one such λ_j exists, and $q_i = 0$ otherwise. Define $q := \max\{q_i : 1 \leq i \leq d\}$. Then (2) has for n sufficiently large a fundamental matrix satisfying*

$$Y_n = [I + O(n^p[q^n + \eta^n])] \, A^n \quad \textit{as} \quad n \to \infty \,, \tag{7}$$

where p can be explicitly estimated.