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Quasi-Lie families, schemes, invariants and their applications to Abel equations

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ABSTRACT

This work analyses types of group actions on families of t-dependent vector fields of a particular class, the hereby called *quasi-Lie families*. We devise methods to obtain the defined here *quasi-Lie invariants*, namely a kind of functions constant along the orbits of the above-mentioned actions. Our techniques lead to a deep geometrical understanding of quasi-Lie schemes and quasi-Lie systems giving rise to several new results. Our achievements are illustrated by studying Abel and Riccati equations. We retrieve the Liouville invariant and study other new quasi-Lie invariants of Abel equations. Several Abel equations with a superposition rule are described and we characterise Abel equations via quasi-Lie schemes.

1. Introduction

The theory of Lie systems investigates a class of non-autonomous systems of first-order ordinary differential equations, the *Lie systems*, admitting a *superposition rule*, i.e. a function permitting us to describe the general solution of such a system in terms of a generic family of particular solutions and some constants [10,15,35,48]. Lie systems have been attracting attention due to their geometric properties and applications [8,10].

Despite providing useful geometric methods to study Lie systems and possessing many applications, the theory of Lie systems presents an important drawback: just a few, but important, systems of differential equations can be studied through this theory [10,31]. To employ the theory of Lie systems for analysing more general systems, the notion of *quasi-Lie scheme* emerged [12]. This concept led to the description of new and known properties of non-linear oscillators [12], second-order Gambier equations [16], Emden–Fowler

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equations [17], second-order Riccati equations [9], dissipative Milne–Pinney equations [10], Abel equations [11], etc.

Roughly speaking, quasi-Lie schemes describe the transformation properties of certain collections of non-autonomous systems of first-order ordinary differential equations of a particular kind, the so-called *quasi-Lie families*, under the action of a related group of t-dependent changes of variables [12, Theorem 1]. As a simple application, this enables us to map Abel equations onto Abel equations providing methods to integrate them [11].

Many procedures developed to study the integrability of a non-autonomous system of differential equations of a certain sort, e.g. Abel equations, rely on mapping it onto an autonomous and easily integrable one of the same type by transformations generally provided in an *ad hoc* way. Meanwhile, the theory of quasi-Lie schemes may naturally provide answers to these questions [11], e.g. it justifies geometrically the group of transformations employed to study Abel equations [11]. Since such answers depend only on the algebraic/geometric structure of the system of differential equations under study, the so found properties can easily be generalised to other systems of differential equations with a similar structure.

As a first new result, we succeed in characterising groups of t-dependent transformations mapping elements of a quasi-Lie family onto other (possibly the same) members of the quasi-Lie family: the so-called *structure preserving groups* of the quasi-Lie family. The elements of such a group of transformations, the Lie almost symmetries, can be considered as a generalisation of the well-known Lie symmetries. Our new theory recovers as particular cases the (extended) quasi-Lie schemes groups for quasi-Lie schemes described in [12,17]. We also succeeded in describing the whole family of t-dependent transformations mapping Abel equations onto Abel equations adding details not discussed in [11].

Next, we characterise Abel equations as elements of the quasi-Lie family associated with a quasi-Lie scheme. This shows that many of the properties of Abel equations are essentially algebraic and geometric and are due to the structure of a quasi-Lie scheme. This motivates to define new structures related to quasi-Lie schemes: quasi-Lie schemes morphisms, the representation associated with a quasi-Lie scheme, etc. On the one hand, these notions enlarge the theoretical content of the theory of quasi-Lie schemes. On the other hand, we generalise the so-called hierarchies of Lie systems given in [5] and we extrapolate the results on integrability of Abel equations to other equations, e.g. Lotka–Volterra systems, and vice versa.

Another important question is whether we can connect two elements of a quasi-Lie family by means of a Lie almost symmetry. To provide necessary conditions for the existence of such a Lie almost symmetry, we define the *quasi-Lie invariants* of a quasi-Lie family, i.e. functions mapping each element of the quasi-Lie family onto a t-dependent function that is the same for all elements connected by a Lie almost symmetry.

We describe and analyse different types of quasi-Lie invariants and we characterise them in geometrical and algebraic terms. As an application, we retrieve the known Liouville invariant of Abel equations and we prove the existence of other new quasi-Lie invariants.

The study of Abel equations is currently a very active field of research that aims, for example, to determinate their integrability conditions, exact solutions, and general properties [3,22,33,36,41,44–46,49]. For instance, Lie symmetries of Abel equations were studied in [32]. Their solutions via transformations $y = u(x)z(x) + v(x)$ were investigated in [42] and through the referred to as Julia's condition in [6]; equivalence and integrable cases are presented in [21]; etc. Meanwhile, some solutions in closed form appeared in [39] and periodic solutions have been studied in [2]. Abel equations are also relevant for other mathematical problems like the centre problem for planar polynomial systems of differential equations [4,7,26,27,40]. Moreover, Abel equations are broadly used in Physics, where they describe the properties of a number of interesting physical systems [19,23–25,29,38,41,44,51]. For instance, Abel equations play a role in the analysis of cosmological models [28,29,50] and the theory of membranes in M-theory [23,51]. Hence, the analysis of Abel equations is very relevant so as to determine the properties of the physical systems they describe.