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On a transformation between distributions obeying the principle of a single big jump ☆

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ABSTRACT

Beck et al. (2013) introduced a new distribution class $\mathcal{J}$ which contains many heavy-tailed and light-tailed distributions obeying the principle of a single big jump. Using a simple transformation which maps heavy-tailed distributions to light-tailed ones, we find some light-tailed distributions, which belong to the class $\mathcal{J}$ but do not belong to the convolution equivalent distribution class and which are not even weakly tail equivalent to any convolution equivalent distribution. This fact helps to understand the structure of the light-tailed distributions in the class $\mathcal{J}$ and leads to a negative answer to an open question raised by the above paper.

1. Introduction

Recently, Beck et al. [1] introduced a new distribution class in which all distributions obey the *principle of a single big jump*.

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution F. Define the class $\mathcal{J}$ as the set of all distributions F with unbounded support contained in $[0, \infty)$ such that for all $n \geq 2$ (or, equivalently, for $n = 2$),

$$\lim_{K\to\infty} \liminf_{x\to\infty} \mathbb{P}\big(X_{n,1} > x - K \mid S_n > x\big) = 1, \tag{1.1}$$

where $X_{n,k}$ means the k-th largest random variable in the sequence $\{X_i, 1 \leq i \leq n\}$, $1 \leq k \leq n$, and $S_n = \sum_{i=1}^{n} X_i$. Eq. (1.1) says that conditionally on S_n being unusually large, the largest summand $X_{n,1}$ is likely to be almost as large as S_n. As shown in [1], (1.1) is equivalent to saying that the family of conditional laws of $X_{n,2}$ conditioned on $S_n > x$ is tight for $x > 0$ which is one way of defining what it means that a distribution F obeys the principle of a single big jump.

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Further, [1] includes a systematic study of properties of the class $\mathcal{J}$ such as its closure under weak tail equivalence with applications to random walks and infinitely divisible distributions. Xu et al. [22] construct some new examples for heavy-tailed distributions in the class $\mathcal{J}$ to explain the differences between the class $\mathcal{J}$ and some well-established distribution classes, and describe the structure of the heavy-tailed distributions in the class $\mathcal{J}$. By contrast, using a simple transformation which maps heavy-tailed to light-tailed distributions, we will illuminate the structure of the light-tailed distributions in $\mathcal{J}$ in the present paper. The structure is related to the convolution equivalent distribution class which, along with several heavy-tailed classes, will be introduced below.

In this paper, unless otherwise stated, F is a distribution with unbounded support in $[0, \infty)$ with tail $\overline{F} := 1 - F$, $F * G$ is the convolution of two distributions F and G, and $F^{*i} := F * \cdots * F$ is the i-fold convolution of F with itself for $i \geq 2$. In addition, all limits and asymptotic equivalences refer to x tending to infinity.

Recall that a distribution F belongs to the *subexponential* distribution class $\mathcal{S}$ introduced by Chistyakov [2], if

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x),$$

where the notation $g(x) \sim h(x)$ means that the ratio $g(x)(h(x))^{-1} \to 1$ for two positive functions g and h supported on $[0, \infty)$. The class $\mathcal{S}$ is properly included in the following larger distribution class.

We say that a distribution F belongs to the long-tailed distribution class $\mathcal{L}$, if for all (or equivalently, for some) constant $t \neq 0$

$$\overline{F}(x + t) \sim \overline{F}(x),$$

see, for example, [2]. The class $\mathcal{L}$ is an important subclass of the heavy-tailed distribution class $\mathcal{K}$ satisfying $\int_0^\infty e^{\lambda y} F(dy) = \infty$ for all $\lambda > 0$. We will denote its complement, the class of light-tailed distributions, by $\mathcal{K}^c$.

The following heavy-tailed distribution subclass was introduced by Feller [8]. A distribution F belongs to the dominatedly-varying-tailed distribution class, denoted by $\mathcal{D}$, if for all (or equivalently, for some) $t \in (0, 1)$

$$\overline{F}(tx) = O(\overline{F}(x)),$$

where the notation $g(x) = O(h(x))$ means that $\limsup g(x)(h(x))^{-1} < \infty$ for two positive functions g and h supported on $[0, \infty)$. The classes $\mathcal{L}$ and $\mathcal{D}$ are not comparable, see, for example, Embrechts et al. [7].

It is proved in [1] that $\mathcal{L} \cap \mathcal{J} = \mathcal{S}$ and $\mathcal{S} \cup \mathcal{D} \subset \mathcal{J}$. The last inclusion relationship is proper, see [22]. In addition, the class $\mathcal{J}$ contains some common light-tailed distributions.

We say that a distribution F belongs to the exponential distribution class with parameter $\gamma > 0$, denoted by $\mathcal{L}(\gamma)$, if for all $t \neq 0$

$$\overline{F}(x + t) \sim e^{-\gamma t}\overline{F}(x),$$

see, for example, Chover et al. [3,4]. Further, if $F \in \mathcal{L}(\gamma)$ for some constant $\gamma > 0$, $\int_0^\infty e^{\gamma y} F(dy) < \infty$ and

$$\overline{F^{*2}}(x) \sim 2\int_0^\infty e^{\gamma y} F(dy)\overline{F}(x),$$

then we say that F belongs to the convolution equivalent distribution class, denoted by $\mathcal{S}(\gamma)$ which was introduced in [3,4].