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# Conservation laws for some equations that admit compacton solutions induced by a non-convex convection

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ABSTRACT

In this study, we consider some equations that admit compacton solutions induced by a non-convex convection. Some conservation laws are derived for the nonlinearly self-adjoint equations based on differential substitutions. We also derive some conservation laws using the multipliers method.



## 1. Introduction

It is well known that solitons have utility in many physical processes. However, solitons have many obvious shortcomings, such as their infinite tails and considerably reduced ubiquity in higher spatial dimensions because the impact of nonlinearity typically changes little with the spatial dimension, and thus the spreading effectiveness of dispersive mechanisms increases. To address both of these issues, the compactons, i.e., solitons with a compact support, were introduced by Rosenau and Hyman over 20 years ago based on the K$(m,n)$ model equation,

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1,$$

[19].

It is known that conservation laws play a significant role in the solution of an equation or a system of differential equations. Not all of the conservation laws of partial differential equations (PDEs) have physical interpretations, but they are essential for studying the integrability of PDEs. Moreover, the conservation laws are used for analysis, particularly the development of numerical schemes and the study of properties such as bi-Hamiltonian structures and recursion operators, as well as the reduction of PDEs. For variational problems, the Noether theorem [1] can be used to derive the conservation laws. For non-variational problems,

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there are different methods for constructing the conservation laws. In [1], Anco and Bluman gave a general treatment of a direct conservation law method for PDEs expressed in a standard Cauchy–Kovalevskaya form. In [12], a general theorem that does not require the existence of Lagrangians was introduced by Ibragimov. This theorem is based on the concept of adjoint equations for nonlinear equations. The concept of strictly self-adjoint equations has also been generalized [13,7,15]. In [14,15], a definition was introduced for a nonlinearly self-adjoint equation, where the substitution can involve the dependent variable as well as its derivatives and the independent variables. After Ibragimov, several studies addressed self-adjointness and its applications to PDEs [2,4,3,5,16,17,8,10,9]. Recently, the property of nonlinear self-adjointness with differential substitution was considered in [11]. In [22], conditions were given for the existence of multipliers involving derivatives, the existence of which is connected to the nonlinear self-adjointness with differential substitution for the evolution of PDEs.

The conservation laws for the compacton K$(2,2)$ equation and the compacton K$(3,3)$ equation were constructed in [18] utilizing the multipliers approach. In [18], only multipliers of the form $A(t,x,u)$ were considered because, according to the authors, the higher order multipliers that determine equations are too complicated and they cannot be separated manually. In a recent study [20], Rosenau and Oron explored the impact of two prototypical types of a non-convex convection on a formation of compact patterns, which was achieved using a simple model

$$u_t + k(u^3 - u^2)_x + (u^3)_{xxx} = 0, \tag{1}$$

with $k = \pm 1$. In contrast to the K$(n,n)$-compactons that have a universal, speed-independent width, they found that the width of the proposed compactons depends on their velocity. The assumed non-convex convection may be due to two opposing convection forces, such as those that can occur in a liquid layer on a tilted plane when gravity and Marangoni forces induce fluid convection, but in opposite directions. In [19], it was noted that it is natural to examine the impact of a non-convex convection on familiar grounds in the KdV universe:

$$u_t + k(u^3 - u^2)_x + u_{xxx} = 0, \tag{2}$$

with $k = \pm 1$. In [21], a quasi self-adjoint classification was provided for a general class of evolution equations, which includes the KdV equation, Eqs. (1) and (2), as well as others. In [5], the authors considered a class of evolution equations up to fifth-order that contain many arbitrary smooth functions from the viewpoint of nonlinear self-adjointness. However, only a few studies have employed substitution depending on the derivatives, e.g., [6,11]. Therefore, in this study, we prove that Eqs. (1) and (2) are both nonlinearly self-adjoint, and using the notations and techniques in [12], we also determine some non-trivial conservation laws for Eqs. (1) and (2) using the multipliers method.

## 2. Nonlinearly self-adjoint equations

Now, using explicit constructions, we prove that Eqs. (1) and (2) are nonlinearly self-adjoint in the sense of [15].

Eqs. (1) and (2) are written in the form

$$F = 0 \tag{3}$$

with

$$F = u_t + k(u^3 - u^2)_x + (u^3)_{xxx}$$