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Existence results for a nonlinear transmission problem

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ABSTRACT

Let Ω^o and Ω^i be open bounded regular subsets of $\mathbb{R}^n$ such that the closure of Ω^i is contained in Ω^o. Let f^o be a regular function on $\partial\Omega^o$ and let F and G be continuous functions from $\partial\Omega^i \times \mathbb{R}$ to $\mathbb{R}$. By exploiting an argument based on potential theory and on the Leray–Schauder principle we show that under suitable and completely explicit conditions on F and G there exists at least one pair of continuous functions (u^o, u^i) such that

$$\begin{cases} \Delta u^o = 0 & \text{in } \Omega^o \setminus \operatorname{cl}\Omega^i, \\ \Delta u^i = 0 & \text{in } \Omega^i, \\ u^o(x) = f^o(x) & \text{for all } x \in \partial\Omega^o, \\ u^o(x) = F(x, u^i(x)) & \text{for all } x \in \partial\Omega^i, \\ \nu_{\Omega^i} \cdot \nabla u^o(x) - \nu_{\Omega^i} \cdot \nabla u^i(x) = G(x, u^i(x)) & \text{for all } x \in \partial\Omega^i, \end{cases}$$

where the last equality is attained in certain weak sense. A simple example shows that such a pair of functions (u^o, u^i) is in general neither unique nor locally unique. If instead the fourth condition of the problem is obtained by a small nonlinear perturbation of a homogeneous linear condition, then we prove the existence of at least one classical solution which is in addition locally unique.

1. Introduction

We investigate the existence of solutions for a boundary value problem with a nonlinear transmission condition. In order to define such a boundary value problem we introduce some notation. We fix once for all

$$\text{a natural number } n \in \mathbb{N},\ n \geq 2, \text{ and a real number } \alpha \in]0, 1[,$$

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where $\mathbb{N}$ denotes the set of natural numbers including 0. Then we fix two sets Ω^o and Ω^i in the n-dimensional Euclidean space $\mathbb{R}^n$. The letter 'o' stands for 'outer domain' and the letter 'i' stands for 'inner domain'. We assume that Ω^o and Ω^i satisfy the following condition:

$$\Omega^o \text{ and } \Omega^i \text{ are open bounded subsets of } \mathbb{R}^n \text{ of class } C^{1,\alpha}, \ \operatorname{cl}\Omega^i \subseteq \Omega^o, \text{ and}$$
$$\text{the boundaries } \partial\Omega^o \text{ and } \partial\Omega^i \text{ are connected.}$$

For the definition of functions and sets of the usual Schauder class $C^{0,\alpha}$ and $C^{1,\alpha}$, we refer for example to Gilbarg and Trudinger [20, §6.2]. Here and in the sequel $\operatorname{cl}\Omega$ denotes the closure of Ω for all $\Omega \subseteq \mathbb{R}^n$. Then we fix a function $f^o \in C^{1,\alpha}(\partial\Omega^o)$ and two continuous functions F and G from $\partial\Omega^i \times \mathbb{R}$ to $\mathbb{R}$ and we consider the following nonlinear transmission boundary value problem for a pair of functions (u^o, u^i) in $C^{1,\alpha}(\operatorname{cl}\Omega^o \setminus \Omega^i) \times C^{1,\alpha}(\operatorname{cl}\Omega^i)$,

$$\begin{cases} \Delta u^o = 0 & \text{in } \Omega^o \setminus \operatorname{cl}\Omega^i, \\ \Delta u^i = 0 & \text{in } \Omega^i, \\ u^o(x) = f^o(x) & \text{for all } x \in \partial\Omega^o, \\ u^o(x) = F(x, u^i(x)) & \text{for all } x \in \partial\Omega^i, \\ \nu_{\Omega^i} \cdot \nabla u^o(x) - \nu_{\Omega^i} \cdot \nabla u^i(x) = G(x, u^i(x)) & \text{for all } x \in \partial\Omega^i, \end{cases} \tag{1}$$

where ν_{Ω^i} denotes the outer unit normal to the boundary $\partial\Omega^i$. Our aim is to determine suitably general and completely explicit conditions on F and G which ensure the existence of solutions of problem (1).

The analysis of problems such as (1) is motivated by the role played in continuum mechanics. In particular, nonlinear transmission conditions of this kind arise in the study of composite structures glued together by thin adhesive layers which are thermally or mechanically very different from the components' constituents. In modern material technology such composites are widely used (see, *e.g.*, the second named author, Miszuris, and Öchsner [34,35] and Rosselli and Carbutt [40]), but the numerical treatment of the mathematical model by finite elements methods is still difficult, requires the introduction of highly inhomogeneous meshes, and often leads to poor accuracy and numerical instability (see, *e.g.*, Babuška and Suri [1]). A convenient way to overcome this problem is to replace the thin layers by zero thickness interfaces between the composite's components. Then one has to define on such interfaces suitable transmission conditions which incorporate the thermal and mechanical properties of the original layers. Such a procedure can be rigorously justified by an asymptotic method and leads to the introduction of boundary value problems with nonlinear transmission conditions such as those in (1) (see for example Miszuris and Öchsner [36] and the references therein).

We observe that the existence of solutions of nonlinear boundary value problems has been largely investigated by means of variational techniques (see, *e.g.*, the monographs of Nečas [39] and of Roubíček [41] and the references therein). In fact, under some restrictive assumptions on the functions F and G, the existence of solutions of our problem (1) could be deduced by exploiting some known results. In particular, if it happens that problem (1) can be reformulated into an equation of the form $-\operatorname{div} A(x, U)\nabla U = 0$, where A is a suitable Carathéodory function and the unknown function U belongs to the Sobolev space $H^1(\Omega^o)$ and satisfies a Dirichlet condition on $\partial\Omega^o$, then the existence and uniqueness of a solution can be directly deduced by the results of Hlaváv̌cek, Křížek and Malý in [21]. This is for example the case when $G = 0$ and the function $F(x, t)$ of $(x, t) \in \partial\Omega^i \times \mathbb{R}$ is constant with respect to x, is differentiable with respect to t, and the partial differential $\partial_t F(x_0, \cdot)$ is Lipschitz continuous and satisfies the inequality $1/c < \partial_t F(x_0, t) < c$ for a constant $c > 0$ and for all $t \in \mathbb{R}$ (here x_0 is a fixed point of $\partial\Omega^i$).

In this paper instead, we exploit a method based on potential theory to rewrite problem (1) into a suitable nonlinear system of integral equations which can be analysed by a fixed-point theorem. Potential theoretic techniques have been largely exploited in literature to study existence and uniqueness problems for linear