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Gap theorems on hypersurfaces in spheres ☆

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ABSTRACT

We study a complete noncompact hypersurface M^n isometrically immersed in an $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$ $(n \geq 3)$. We prove that there is no non-trivial L^2-harmonic 2-form on M, if the length of the second fundamental form is less than a fixed constant. We also showed that the same conclusion holds if the scale-invariant total tracefree curvature is bounded above by a small constant depending only on n. These results are generalized versions of the result of Cheng and Zhou on bounded harmonic functions with finite Dirichlet integral and the one of Fang and the author on L^2 harmonic 1-forms.

1. Introduction

It is well known that harmonic forms give canonical representation to de Rham cohomology on compact manifolds. Suppose that (M^n, g) is a complete Riemannian manifold of dimension n $(n \geq 3)$. Let $\{e_1, \ldots, e_n\}$ be locally defined orthogonal frame fields of tangent bundle TM. Denote the dual coframe fields by $\{e^1, \ldots, e^n\}$. Suppose

$$\omega = a_{i_1 \ldots i_p} e^{i_p} \wedge \cdots \wedge e^{i_1} = a_I \omega_I,$$

$$\theta = b_{i_1 \ldots i_p} e^{i_p} \wedge \cdots \wedge e^{i_1} = b_I \omega_I,$$

where the summation is being performed over the multi-index $I = (i_1, \ldots, i_p)$, $a_{i_1 \ldots i_k \ldots i_l \ldots i_p} = -a_{i_1 \ldots i_l \ldots i_k \ldots i_p}$ and $b_{i_1 \ldots i_k \ldots i_l \ldots i_p} = -b_{i_1 \ldots i_l \ldots i_k \ldots i_p}$. Set

$$|\omega|^2 = \sum_I a_I^2,$$

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$$|\nabla\omega|^2 = \sum_{i=1}^{n} |\nabla_{e_i}\omega|^2$$

and

$$\langle\omega,\theta\rangle = \sum_I a_I b_I.$$

The space $L^2(\wedge^p T^*M)$ is a Hilbert space with scalar product:

$$(\omega,\theta) = \int_M \langle\omega,\theta\rangle d\mu.$$

Recall that Hodge operator $* : \wedge^p T^*M \to \wedge^{(n-p)} T^*M$ is determined by

$$*e^{i_1} \wedge \cdots \wedge e^{i_p} = \operatorname{sgn}\sigma(i_1 \dots i_p, i_{p+1} \dots i_n) e^{i_{p+1}} \wedge \cdots \wedge e^{i_n},$$

where $\sigma(i_1 \dots i_p, i_{p+1} \dots i_n)$ denotes a permutation of the set $(i_1 \dots i_p, i_{p+1} \dots i_n)$ and $\operatorname{sgn}\sigma$ is the sign of σ. The operator $d^* : \wedge^p T^*M \to \wedge^{(p-1)} T^*M$ is defined by

$$d^*\omega = (-1)^{np+n+1} * d * \omega.$$

The Laplacian operator on M is defined by

$$\triangle\omega = -dd^* - d^*d.$$

A p-form ω is called harmonic if $\triangle\omega = 0$. A p-form ω is called L^2 harmonic if $\omega = 0$ is harmonic and $\omega \in L^2(\wedge^p T^*M)$. A p-form ω is called parallel if $\nabla\omega = 0$, where ∇ is the Levi-Civita connection of (M, g). Each L^2 harmonic p-form is closed and coclosed [14]. We denote by $H^p(L^2(M))$ the space of all L^2 harmonic p-forms. Let

$$Z_2^p(M) = \{\alpha \in L^2(\wedge^p(T^*M)) : d\alpha = 0\}$$

and

$$D^p(d) = \{\alpha \in L^2(\wedge^p(T^*M)) : d\alpha \in L^2(\wedge^{p+1}(T^*M))\}.$$

We define the p-th space of reduced L^2 cohomology by

$$H_2^p(M) = \frac{Z_2^p(M)}{\overline{dD^{p-1}(d)}}.$$

When (M, g) is a complete Riemannian manifold, Carron [1, Corollary 1.6] proved that the space of L^2 harmonic p-forms $H^p(L^2(M))$ is isomorphic to the p-th reduced L^2 cohomology $H_2^p(M)$, for $0 \le p \le n$.

Yau [14] showed that if a complete noncompact manifold M has non-negative Ricci curvature, then there is no non-trivial L^2 harmonic 1-form and volume of M is infinite. Palmer [10] obtained that there exists no non-trivial L^2 harmonic 1-form on a complete noncompact stable minimal hypersurface in $\mathbb{R}^{n+1}$. This result has been generalized by Tanno [12] and Cheng [3] when the ambient space is replaced by a manifold with bi-Ricci curvature having a low bound. In [4,11], it is showed that a complete minimal stable (or weakly stable) hypersurface in $\mathbb{R}^{n+1}$ ($n \ge 3$) with finite total curvature is a hyperplane. Cavalcante, Mirandola and