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# An estimate for the Siciak extremal function – Subanalytic geometry approach

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ABSTRACT

We study the Łojasiewicz–Siciak inequality in $\mathbb{C}^N$. This concerns the estimating from below the Siciak extremal function which is very much related to the Green function in $\mathbb{C}$. As an application we present some results describing how fast holomorphic functions defined in a neighbourhood of a compact holomorphic polyhedron can be approximated uniformly on this polyhedron by complex polynomials.



## 1. Introduction

There are many papers dealing with the problem of estimating from above, near the boundary of a compact set $K \subset \mathbb{C}^N$, the Siciak extremal function $\Phi_K$. Recall that $\Phi_K$ is defined as follows

$$\Phi_K(z) := \sup\{|Q(z)|^{1/\deg Q} : Q \in \mathbb{C}[Z],\ \deg Q > 0 \text{ and } \|Q\|_K \le 1\},$$

for $z \in \mathbb{C}^N$, where $\mathbb{C}[Z] = \mathbb{C}[Z_1, \dots, Z_N]$ is the space of complex polynomials, $\|Q\|_K := \sup_{z\in K} |Q(z)|$.

This function was introduced by J. Siciak and is a very important tool in (pluri)potential theory and approximation theory (see for example [21,24,34,37,38,41]). Recall moreover that if $K \subset \mathbb{C}$ is a compact set with $\mathbb{C} \setminus K$ connected and $\Phi_K$ is continuous, then

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$$\log \Phi_K(z) = g_{\overline{\mathbb{C}}\setminus K}(z,\infty)\,, \quad z \in \mathbb{C}\setminus K\,,$$

where $\overline{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ and $g_{\overline{\mathbb{C}}\setminus K}(z,\infty)$ is the Green function of $\overline{\mathbb{C}}\setminus K$ with pole at $\infty$.[1]

Since the explicit formula for $\Phi_K$ is unknown in general (it is really unknown even for very simple sets), it is also important to estimate this function from below. In particular, recently there is a growing interest in the so-called Łojasiewicz–Siciak condition introduced in [11]. We say that a compact set $K \subset \mathbb{C}^N$ satisfies the (ŁS) condition (inequality) if it is polynomially convex and there exist constants $\eta > 0$, $\kappa > 0$ such that

$$\Phi_K(z) \geq 1 + \eta\,(\operatorname{dist}(z;\,K))^{\kappa} \quad \text{as } \operatorname{dist}(z;\,K) \leq 1$$

($z \in \mathbb{C}^N$).[2]

Recall that a compact set $K \subset \mathbb{C}^N$ is polynomially convex if $K = \hat{K}$, where

$$\hat{K} := \{z \in \mathbb{C}^N : \; |Q(z)| \leq \|Q\|_K \text{ for all polynomials } Q \in \mathbb{C}[Z]\}\,.$$

It is well known that a compact set $K \subset \mathbb{C}$ is polynomially convex if and only if $\mathbb{C}\setminus K$ is connected – see [13], Corollary 1.3.2.

The interest in the Łojasiewicz–Siciak condition comes also from the fact that it has applications in approximation of functions (see [11] and Sections 6, 9). The problem is to decide, given a compact subset of $\mathbb{C}^N$, whether or not it satisfies the Łojasiewicz–Siciak condition. The partial answers in $\mathbb{C}$ are given in [29]. It is also explained in [29] that there is a great difference between the situation when a set $K \subset \mathbb{R}^2$ is treated as a subset of $\mathbb{C}^2$ and the situation when it is treated as a subset of $\mathbb{C}$. In $\mathbb{C}^N$ for $N \geq 2$, the problem seems to be really difficult. One of the reasons is the lack of suitable tools (available for example in $\mathbb{C}$). Of course we do not speak here about the sets $K \subset \mathbb{C}^N$ for which the function $\Phi_K$ is explicitly computed. However, as we already mentioned, there are very few such examples, especially if $N \geq 2$, and they are described in [21].

In our paper we study the Łojasiewicz–Siciak inequality from the point of view of subanalytic geometry developed in the seminal works of S. Łojasiewicz, H. Hironaka and many other researchers (cf. [6,12,25]). Let us mention that there are also other topics in potential and pluripotential theory that have been studied in the setting of subanalytic geometry (or o-minimal geometry which is a natural generalization of subanalytic geometry) – see for example the papers by T. Kaiser, W. Pawłucki, W. Pleśniak or the author [14–20,26–28, 33,35].

The main aims of the paper are:

- to give new examples of sets in $\mathbb{C}^N$ satisfying the Łojasiewicz–Siciak condition (see Theorem 3.1 and Section 4),
- to show that this has applications in approximation of holomorphic functions (see Theorems 6.1, 6.3, 6.4 and 9.1),
- to discuss the connections of the Łojasiewicz–Siciak condition and the (HCP) condition with the problem of the definability of the extremal function (see Section 5).

## 2. On the polynomial convexity assumption

At the beginning of our paper we will show in particular that the polynomial convexity assumption in the definition of the Łojasiewicz–Siciak condition posed in [5,11] is superfluous. Namely, we have the following result.

[1] Generally, $g_\Omega(z, z_0)$ denotes the Green function of $\Omega$ with pole at $z_0$.
[2] In $\mathbb{C}^N$ we consider the Euclidean norm $|\;|$.