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On the large deviation principle of generalized Brownian bridges

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ABSTRACT

In this paper we consider a family of generalized Brownian bridges with a small noise, which was used by Brennan and Schwartz [3] to model the arbitrage profit in stock index futures in the absence of transaction costs. More precisely, we study the large deviation principle of these generalized Brownian bridges as the noise becomes infinitesimal.

1. Introduction

1.1. α-Brownian bridge

Let $\{W_t\}_{t\geq 0}$ be the standard Wiener process. For positive parameters α and ε, the *α-Brownian bridge* $\{X_t^\varepsilon\}_{0\leq t\leq 1}$ starting from $a \in \mathbb{R}$ and ending at zero can be constructed as follows. First, there is a unique strong solution $\{X_t^\varepsilon\}_{0\leq t<1}$ to the following stochastic differential equation (cf. [9])

$$\begin{cases} dX_t^\varepsilon = -\dfrac{\alpha \cdot X_t^\varepsilon}{1-t}dt + \sqrt{\varepsilon}dW_t, \quad 0 \leq t < 1, \\ X_0^\varepsilon = a. \end{cases} \tag{1.1}$$

In this case X_t^ε can be represented as (cf. [7])

$$X_t^\varepsilon = (1-t)^\alpha \left(x + \int_0^t \frac{\sqrt{\varepsilon}}{(1-s)^\alpha} dW_s \right), \quad 0 \leq t < 1.$$

From $\alpha > 0$ it follows that $\lim_{t\to 1} X_t^\varepsilon = 0$ almost surely, which gives the definition of α-Brownian bridge $\{X_t^\varepsilon\}_{0\leq t\leq 1}$ from a position a at $t = 0$ to a position 0 at $t = 1$. We also note that X^ε cannot have an

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almost surely continuous extension to a fixed position at $t = 1$ if $\alpha \leq 0$. This can be intuitively seen by noticing that we have a scaled Wiener process if $\alpha = 0$ and the variance of X_t^ε tends to infinity as $t \to 1$ if $\alpha < 0$. If the parameter $\alpha = 1$, then X^ε becomes a classical Brownian bridge determined by $dX_t^\varepsilon = -\frac{X_t^\varepsilon}{1-t}dt + \sqrt{\varepsilon}dW_t$, $X_0^\varepsilon = a$, or equivalently (in the sense of distribution) $\widetilde{X}_t^\varepsilon = a(1-t) + \sqrt{\varepsilon}(W_t - tW_1)$, $0 \leq t \leq 1$.

The generalized Brownian bridges considered in this paper are the α-Brownian bridges defined as

$$Y_t^\varepsilon = (1-t)^\alpha \left[(a-b) + \int_0^t \frac{\sqrt{\varepsilon}}{(1-s)^\alpha} dW_s \right] + b, \quad 0 \leq t < 1, \tag{1.2}$$

with $\lim_{t\to 1} Y_t^\varepsilon = b$ for $\alpha > 0$. Thus the bridges have an initial position $Y_0^\varepsilon = a$ and a terminal position $Y_1^\varepsilon = b$. The bridge Y^ε is just a shift of size b of the original bridge X^ε with an adjusted initial position $a - b$. In their study of arbitrage in stock index futures with transaction costs and position limits, Brennan and Schwartz [3] suggested that the arbitrage profit associated with a given futures contract follows the evolution of Y^ε. There, the maturity is $T > 0$ (here we choose 1 for simplicity), α determines the speed of mean reversion, and $\sqrt{\varepsilon}$ is the instantaneous standard deviation (see also [3], [13] and [12] for related discussions in finance). The estimations of α and $\sqrt{\varepsilon}$ in [3], based on 16 Standard and Poor's (S&P) 500 Stock Index Futures contracts maturing from September 1983 to June 1987, suggest that $\alpha \geq 1$ and $\sqrt{\varepsilon}$ is small. It is then natural to investigate the convergence of $\{Y_t^\varepsilon\}_{0\leq t\leq 1}$ as the noise $\sqrt{\varepsilon}dW_t$ becomes infinitesimal. We will study this aspect in the sense of large deviations.

1.2. The main result

Let us first define the function space $\mathbb{C}^{a,b}[0,1]$ which will be used throughout the paper

$$\mathbb{C}^{a,b}[0,1] = \{\phi(\cdot) \in \mathbb{C}[0,1] \ : \ \phi(0) = a \text{ and } \phi(1) = b\}.$$

To formulate the large deviations of Y^ε, we define the *rate function* (or *action functional* in the quantum context) as

$$S(\phi) = \frac{1}{2} \int_0^1 \left(\phi'(t) + \frac{\alpha}{1-t}(\phi(t) - b) \right)^2 dt \tag{1.3}$$

for absolutely continuous $\phi \in \mathbb{C}^{a,b}[0,1]$ (otherwise we set $S(\phi) = \infty$).

Theorem 1.1. *Let $\{Y_t^\varepsilon\}_{0\leq t\leq 1}$ be defined as in (1.2) with $\alpha \geq 1$, then $\{Y_t^\varepsilon\}_{0\leq t\leq 1}$ satisfies a large deviation principle with a good rate function $S(\phi)$ given by (1.3). Namely,*

(i) *for any measurable and open set $O \subseteq \mathbb{C}^{a,b}[0,1]$,*

$$\liminf_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}\{Y^\varepsilon \in O\} \geq -\inf_{\phi\in O} S(\phi); \tag{1.4}$$

(ii) *for any measurable and closed set $F \subseteq \mathbb{C}^{a,b}[0,1]$,*

$$\limsup_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}\{Y^\varepsilon \in F\} \leq -\inf_{\phi\in F} S(\phi). \tag{1.5}$$