J. Math. Anal. Appl. 430 (2015) 874–894

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Polynomial and rational inequalities on analytic Jordan arcs and domains

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ARTICLE INFO

Article history:
Received 5 October 2014
Available online 18 May 2015
Submitted by P. Nevai

Dedicated to Professor Vilmos Totik on his sixtieth birthday

Keywords:
Polynomials
Rational functions
Bernstein-type inequalities
Conformal mappings

ABSTRACT

In this paper we prove an asymptotically sharp Bernstein-type inequality for polynomials on analytic Jordan arcs. Also a general statement on mapping of a domain bounded by finitely many Jordan curves onto a complement to a system of the same number of arcs with rational function is presented here. This fact, as well as, Borwein–Erdélyi inequality for derivative of rational functions on the unit circle, Gonchar–Grigorjan estimate of the norm of holomorphic part of meromorphic functions and Totik's construction of fast decreasing polynomials play key roles in the proof of the main result.

0. Introduction

Let $\mathbb{T} := \{z \in \mathbf{C} : |z| = 1\}$ denote the unit circle, $\mathbb{D} := \{z \in \mathbf{C} : |z| < 1\}$ denote the unit disk and $\mathbf{C}_\infty := \mathbf{C} \cup \{\infty\}$ denote the extended complex plane. We also use $\mathbb{D}^* := \{z \in \mathbf{C} : \ |z| > 1\} \cup \{\infty\}$ for the exterior of the unit disk and $\|.\|_K$ for the sup norm over the set K.

First, we recall a Bernstein-type inequality proved by Borwein and Erdélyi in [2] (and in a special case, by Li, Mohapatra and Rodriguez in [7]). We rephrase their inequality using potential theory (namely, normal derivatives of Green's functions) and for the necessary concepts, we refer to [12] and [11]. Then we present one of our main tools, the "open-up" step in Proposition 5, similar step was also discussed by Widom, see [17], pp. 205–206 and Lemma 11.1. This way we switch from polynomials and Jordan arcs to rational functions and Jordan curves. Then we use two conformal mappings, Φ_1 and Φ_2 to map the interior of the Jordan domain onto the unit disk and to map the exterior of the domain onto the exterior

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http://dx.doi.org/10.1016/j.jmaa.2015.05.022
0022-247X/

of the unit disk respectively. We transform our rational function with Φ_1 and "construct" a similar rational function (approximate with another, suitable rational function) so that the Borwein–Erdélyi inequality can be applied.

Our main theorem is the following.

Theorem 1. *Let K be an analytic Jordan arc, $z_0 \in K$ not an endpoint. Denote the two normals to K at z_0 by $n_1(z_0)$ and $n_2(z_0)$. Then for any polynomial P_n of degree n we have*

$$|P_n'(z_0)| \le (1+o(1))\, n \, \|P_n\|_K \cdot \max\left(\frac{\partial}{\partial n_1(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty),\ \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty)\right)$$

where $o(1)$ depends on z_0 and K only and tends to 0 as $n \to \infty$.

Remark. This theorem was formulated as a conjecture in [9] on p. 225.

Theorem 1 is asymptotically sharp as the following theorem shows.

Theorem 2. *Let K be a finite union of disjoint, C^2 smooth Jordan arcs and $z_0 \in K$ is a fixed point which is not an endpoint. We denote the two normals to K at z_0 by $n_1(z_0)$ and $n_2(z_0)$. Then there exists a sequence of polynomials P_n with $\deg P_n = n \to \infty$ such that*

$$|P_n'(z_0)| \ge n(1-o(1))\, \|P_n\|_K \cdot \max\left(\frac{\partial}{\partial n_1(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty), \frac{\partial}{\partial n_2(z_0)} g_{\mathbf{C}_\infty \setminus K}(z_0,\infty)\right).$$

1. A rational inequality on the unit circle

The following theorem was proved in [2] (see also [1], p. 324, Theorem 7.1.7), with slightly different notations.

If f is a rational function, then $\deg(f)$ denotes the maximum of the degrees of the numerator and denominator of f (where we assume that the numerator and the denominator have no common factors).

Theorem *(Borwein–Erdélyi). Let $a_1, \ldots, a_m \in \mathbf{C} \setminus \{|u| = 1\}$ and let*

$$B_m^+(u) := \sum_{j:\ |a_j|>1} \frac{|a_j|^2-1}{|a_j-u|^2}, \qquad B_m^-(u) := \sum_{j:\ |a_j|<1} \frac{1-|a_j|^2}{|a_j-u|^2},$$

and $B_m(u) := \max(B_m^+(u), B_m^-(u))$. If R is a polynomial with $\deg(R) \le m$ and $f(u) = R(u) / \prod_{j=1}^m (u-a_j)$ is a rational function, then

$$|f'(u)| \le B_m(u)\, ||f||_{\mathbb{T}}, \qquad u \in \mathbb{T}.$$

If all the poles of f are inside or outside of $\mathbb{D}$, then this result was improved in [7], Theorem 2 and Corollary 2 on p. 525 using different approach.

We need to relax the condition on the degree of the numerator and the denominator.

If we could allow poles at infinity, then the degree of the numerator can be larger than that of the denominator. More precisely, we can easily obtain the following