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Indefinite least-squares problems and pseudo-regularity

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A R T I C L E I N F O

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A B S T R A C T

Given two Krein spaces $\mathcal{H}$ and $\mathcal{K}$, a (bounded) closed-range operator $C : \mathcal{H} \to \mathcal{K}$ and a vector $y \in \mathcal{K}$, the indefinite least-squares problem consists in finding those vectors $u \in \mathcal{H}$ such that

$$[Cu - y, Cu - y] = \min_{x \in \mathcal{H}} [Cx - y, Cx - y].$$

The indefinite least-squares problem has been thoroughly studied before under the assumption that the range of C is a uniformly J-positive subspace of $\mathcal{K}$. Along this article the range of C is only supposed to be a J-nonnegative pseudo-regular subspace of $\mathcal{K}$. This work is devoted to present a description for the set of solutions of this abstract problem in terms of the family of J-normal projections onto the range of C.

1. Introduction

In signal processing applications it is frequently assumed that the mathematical model, describing the physical phenomena under study, satisfies the following equation:

$$z = Hx + \eta,$$

where $H \in \mathbb{R}^{m\times n}$ is known and $x \in \mathbb{R}^n$ is a parameter that needs to be determined. Sometimes, due to physical restrictions, it is not possible to measure x, and it is necessary to estimate this vector based on

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the measurement z, which is corrupted by noise η. According to the characteristics of the noise, different techniques may be used to estimate x. For instance, when no statistical information about the noise measurement is available, the $\mathcal{H}^\infty$-estimation technique has been proved to be an appropriate approach for several engineering problems. Given $\gamma > 0$, the $\mathcal{H}^\infty$-estimation technique in $\mathbb{R}^n$ consists in finding an estimation $\hat{x}$ of the vector x, such that:

$$\max_{x\in\mathbb{R}^n} \frac{\|x-\hat{x}\|^2}{\|z-Hx\|^2} \leq \gamma^2, \tag{1.1}$$

or equivalently,

$$\min_{x\in\mathbb{R}^n} \left(\|z-Hx\|^2 - \frac{1}{\gamma^2}\|x-\hat{x}\|^2 \right) \geq 0. \tag{1.2}$$

Note that the left hand side of (1.2) can be modeled as the minimization of an indefinite inner product on an affine manifold. In fact, $\mathbb{R}^{m+n}$ can be endowed with the indefinite inner product $[\,x,y\,] := x^T J y$, $x, y \in \mathbb{R}^{m+n}$, where $J \in L(\mathbb{R}^{m+n})$ is the fundamental symmetry given by $J = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$. Then, considering $C := \begin{pmatrix} H \\ \gamma^{-1} I_n \end{pmatrix} \in L(\mathbb{R}^n, \mathbb{R}^{m+n})$ and $y := \begin{pmatrix} z \\ \gamma^{-1}\hat{x} \end{pmatrix} \in \mathbb{R}^{m+n}$, the $\mathcal{H}^\infty$-estimation problem is equivalent to finding a vector y (which depends on z) such that the following indefinite least-squares problem (ILSP) admits a solution:

$$\min_{x\in\mathbb{R}^n} [\,y-Cx, y-Cx\,], \tag{1.3}$$

and to show that this minimum is nonnegative, see [8].

This work is devoted to studying an abstract ILSP: Given arbitrary Krein spaces $\mathcal{H}$ and $\mathcal{K}$, a closed-range operator $C \in L(\mathcal{H},\mathcal{K})$ and a vector $y \in \mathcal{K}$, find the vectors $u \in \mathcal{H}$ such that

$$[\,y-Cu, y-Cu\,] = \min_{x\in\mathcal{H}} [\,y-Cx, y-Cx\,].$$

In finite-dimensional spaces, the ILSP has been exhaustively studied see e.g. [13,14,21,8,15,20,7]. In these papers, if J is the fundamental symmetry of $\mathcal{K}$, it is assumed that $C^T JC$ is a positive-definite matrix, which is a sufficient condition for the existence of a unique solution for the ILSP. This is equivalent to assuming that C is injective and the range of C (hereafter denoted by $R(C)$) is a uniformly J-positive subspace of $\mathcal{K}$. Then, the regularity of $R(C)$ plays an essential role, since it guarantees the existence of a J-selfadjoint projection onto $R(C)$, which determines the unique solution of the ILS problem (1.3).

Even for the general setting it is known that the ILSP admits a solution if and only if $R(C)$ is J-nonnegative and $y \in R(C) + R(C)^{[\perp]}$, see e.g. [6, Thm. 8.4]. Then, the ILSP is well-posed only for the vectors y in the (not necessarily closed) subspace $R(C)+R(C)^{[\perp]}$. Moreover, given $y \in R(C)+R(C)^{[\perp]}$, $u \in \mathcal{H}$ is a solution of the ILSP if and only if $y - Cu \in R(C)^{[\perp]}$ (see Lemma 3.1), i.e. if u is a solution of the normal equation associated to $Cx = y$:

$$C^{\#}(Cx-y) = 0,$$

where $C^{\#}$ stands for the J-adjoint operator of C.

The assumption that $R(C)$ is a uniformly J-positive subspace of $\mathcal{K}$ implies that the ILSP is properly defined for every $y \in \mathcal{K}$, but this is a quite restrictive condition. Along this article (most of the time) it is assumed that $R(C)$ is a J-nonnegative pseudo-regular subspace of $\mathcal{K}$. Thus, the ILSP admits solutions for