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Stability analysis for Magnetic Resonance Elastography ☆

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A R T I C L E I N F O

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A B S T R A C T

We consider the inverse problem of finding unknown elastic parameters from internal measurements of displacement fields for tissues. The measurements are made on the entirety of a smooth domain. Since tissues can be modeled as quasi-incompressible fluids, we examine the Stokes system and consider only the recovery of shear modulus distributions. Our main result is to establish Lipschitz stable estimates on the shear modulus distributions from internal measurements of displacement fields. These estimates imply convergence of a numerical scheme known as the Landweber iteration scheme for reconstructing the shear modulus distributions.

1. Introduction

We consider the problem of Magnetic Resonance Elastography (MRE). MRE aims at providing a quantitative visualization of mechanical properties of human tissues by using the relation between the internal elastic displacement field and the tissue mechanical properties. The mechanical properties of tissue include shear modulus, shear viscosity and compression modulus. Quantification of tissue shear modulus in vivo can provide evidence of the manifestation of tissue diseases. Using internal measurements by MRI scanner of time-harmonic displacement fields offers the possibility of a high-resolved reconstruction of shear modulus distributions, in MRE; see [9]. High resolution is important in the detection of cancerous anomalies in their early stages [2].

In this paper, we provide stability estimates for reconstructing the shear modulus from internal measurements of displacement fields. For doing so, we first reduce the time-harmonic elasticity system to the Stokes system and neglect for simplicity the viscosity. Then we will follow the general approach in [12]. Our other main references are [18,20,24,29]. See [3,4,9,16,22,25,23,28] for recent works on the inverse problem

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in MRE. For recent books and reviews on other inverse problems from internal measurements we refer to [2,17,21,23].

From a mathematical standpoint, inverse problems from internal measurements typically involve more measurements than parameters of interest and allow the recovery of the interesting parameters from an often redundant system of partial differential equations. Many times exact algebraic inversions may not be available. This paper addresses the specific problem of the Stokes system. We reduce the problem of finding the shear modulus to the inversion of an over-determined system of partial differential equations. We prove that under certain hypotheses, this system is elliptic and satisfies the Shapiro–Lopatinskii condition. Moreover, we show that in three dimensions additional internal measurements are needed. For other stability results on inverse problems from internal data for scalar equations we refer the reader to [5,6,11,15,18,27].

Ultimately the goal of examining Stokes system is their use in MRE to detect cancerous anomalies. We use the stability analysis of elliptic systems to prove the convergence of a numerical scheme known as the Landweber iteration scheme. This iteration scheme has already shown success in other simpler models [7].

The paper is organized as follows. In Section 2 we derive the Stokes system from the elasticity equations as the compressional modulus goes to infinity. In Section 3 we provide some preliminary results on over-determined systems of partial differential equations. Section 4 is to prove the stability result of reconstructing the shear modulus from internal measurements of the displacement field in three dimensions. In Section 5, the two-dimensional case is considered. In Section 6 we state the convergence result of the corresponding Landweber scheme. The paper ends with some concluding remarks.

2. Derivation of Stokes system from elasticity equations

Let Ω denote a simply-connected compact and smooth domain in $\mathbb{R}^d$ where $d = 2, 3$ with $\mathcal{C}^\infty$-boundary $\partial\Omega$. We consider

$$f(x) = (f_1(x), f_2(x), \dots, f_d(x)) : \Omega \to \mathbb{R}^d.$$

We use the Einstein summation convention for the rest of this paper.

For two matrices A and B, we let

$$A : B = a_{ij} b_{ij}.$$

We define the Hilbert spaces $H^m(\Omega)^d$ for $m \in \mathbb{N}$, as the completion of the space of $f(x) \in \mathcal{C}_c^\infty(\Omega)^d$ such that

$$\sum_{|i|=1}^{m} \int_\Omega \left(\nabla^i f(x) : \nabla^i f(x) + |f(x)|^2\right) dx < \infty,$$

where $\nabla^i = \partial^{i_1} \dots \partial^{i_d}$ for $i = (i_1, \dots, i_d)$ denote high-order derivatives. We write $|\nabla f|^2 = \nabla f : \nabla f$ from now on. For any $u \in H^1(\Omega)^d$, we let

$$2\nabla^s u = \nabla u + (\nabla u)^t,$$

where ∇u is the matrix $(\partial_j u_i)_{i,j=1}^d$ with u_i as the i-th component of u, and the superscript t denotes the transpose. Let $\mu(x) \in \mathcal{C}^1(\Omega)$, then we define the conormal derivative

$$2\frac{\partial u}{\partial \nu} = \mu(x) \left(\nabla u + (\nabla u)^t\right) n,$$

where n is the outward unit normal to the boundary $\partial\Omega$.