J. Math. Anal. Appl. 430 (2015) 932–949

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A note on a new ideal

Adam Kwela [1]

Institute of Mathematics, University of Gdańsk, Wita Stwosza 58, 80-952 Gdańsk, Poland

A R T I C L E   I N F O

Article history:
Received 26 March 2014
Available online 22 May 2015
Submitted by M. Laczkovich

Keywords:
Ideals
Weakly Ramsey ideals
Ideal convergence
Infinite games
Quasi-continuous functions

A B S T R A C T

In this paper we study a new ideal $\mathcal{WR}$. The main result is the following: an ideal is not weakly Ramsey if and only if it is above $\mathcal{WR}$ in the Katětov order. Weak Ramseyness was introduced by Laflamme in order to characterize winning strategies in a certain game. We apply result of Natkaniec and Szuca to conclude that $\mathcal{WR}$ is critical for ideal convergence of sequences of quasi-continuous functions. We study further combinatorial properties of $\mathcal{WR}$ and weak Ramseyness. Answering a question of Filipów et al. we show that $\mathcal{WR}$ is not 2-Ramsey, but every ideal on ω isomorphic to $\mathcal{WR}$ is Mon (every sequence of reals contains a monotone subsequence indexed by an $\mathcal{I}$-positive set).

1. Introduction

A collection $\mathcal{I} \subset \mathcal{P}(X)$ is an *ideal on* X if it is closed under finite unions and subsets. We additionally assume that $\mathcal{P}(X)$ is not an ideal and each ideal contains $\mathbf{Fin} = [X]^{<\omega}$. In this paper X will always be a countable set. Ideal is *dense* if every infinite set contains an infinite subset belonging to the ideal. The *filter dual to the ideal* $\mathcal{I}$ is the collection $\mathcal{I}^* = \{A \subset X : A^c \in \mathcal{I}\}$ and $\mathcal{I}^+ = \{A \subset X : A \notin \mathcal{I}\}$ is the collection of all $\mathcal{I}$*-positive sets*. If $Y \notin \mathcal{I}$, we can define the restriction of $\mathcal{I}$ to the set Y as $\mathcal{I} \restriction Y = \{A \cap Y : A \in \mathcal{I}\}$. We say that a family $\mathcal{G}$ *generates the ideal* $\mathcal{I}$ if

$$\mathcal{I} = \{A : \exists_{G_0,\dots,G_k \in \mathcal{G}} A \subset G_0 \cup \dots \cup G_k\}.$$

Ideals $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* if there is a bijection $f : \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that

$$A \in \mathcal{I} \Leftrightarrow f^{-1}[A] \in \mathcal{J}.$$

E-mail address: Adam.Kwela@ug.edu.pl.

[1] The author was supported by the NCN (Polish National Science Centre) grant No. 2012/07/N/ST1/03205 and by WCMCS (Warsaw Center of Mathematics and Computer Science).

http://dx.doi.org/10.1016/j.jmaa.2015.05.042
0022-247X/

For simplicity we denote $\sum(i,j) = i+j$ for $(i,j) \in \omega \times \omega$. In the entire paper proj_1 (proj_2) is the projection on the first (second) coordinate, i.e., $\mathrm{proj}_i\colon \omega \times \omega \to \omega$ is given by $\mathrm{proj}_i(x_1, x_2) = x_i$, for $i = 1, 2$.

The structure of ideals on countable sets is often described in terms of orders. We say that $\mathcal{I}$ *is below* $\mathcal{J}$ *in the Katětov order* ($\mathcal{I} \leq_K \mathcal{J}$) if there is $f : \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that

$$A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}.$$

If f is a bijection between $\bigcup \mathcal{J}$ and $\bigcup \mathcal{I}$, we say that $\mathcal{J}$ *contains an isomorphic copy of* $\mathcal{I}$ ($\mathcal{I} \sqsubseteq \mathcal{J}$). Relations between $\leq_K$ and $\sqsubseteq$ were studied in detail in [2]. If $\mathcal{I}$ is a dense ideal, then $\mathcal{I} \sqsubseteq \mathcal{J}$ if and only if there is a 1–1 function $f : \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$ (cf. [2] and [4]).

Ideals $\mathcal{I}$ and $\mathcal{J}$ are $\sqsubseteq$-*equivalent*, if $\mathcal{I} \sqsubseteq \mathcal{J}$ and $\mathcal{J} \sqsubseteq \mathcal{I}$. Obviously, two isomorphic ideals are $\sqsubseteq$-equivalent. The converse does not hold: for instance consider

$$\mathbf{Fin} \otimes \emptyset = \{A \subseteq \omega \times \omega : \{n \in \omega : A_n \neq \emptyset\} \in \mathbf{Fin}\}$$

and

$$\mathcal{P}(\omega) \oplus \mathbf{Fin} = \{A \subseteq \{0,1\} \times \omega : \{n \in \omega : (1,n) \in A\} \in \mathbf{Fin}\}.$$

One can easily see that those ideals are $\sqsubseteq$-equivalent but not isomorphic.

In this paper we introduce a new ideal on $\omega \times \omega$.

Definition 1.1. $\mathcal{WR}$ is an ideal on $\omega \times \omega$ generated by vertical lines (which we call *generators of the first type*) and sets G such that for every $(i,j),(k,l) \in G$ either $i > k+l$ or $k > i+j$ (which we call *generators of the second type*). Equivalently, $\mathcal{WR}$ is generated by homogeneous subsets of the coloring $\lambda\colon [\omega \times \omega]^2 \to 2$ given by:

$$\lambda\left(\{(i,j),(k,l)\}\right) = \begin{cases} 0 & \text{if } k > i+j, \\ 1 & \text{if } k \leq i+j \end{cases}$$

for all (i,j) below (k,l) in the lexicographical order.

The space 2^X of all functions $f : X \to 2$ is equipped with the product topology (each space $2 = \{0,1\}$ carries the discrete topology). We treat $\mathcal{P}(X)$ as the space 2^X by identifying subsets of X with their characteristic functions. All topological and descriptive notion in the context of ideals on X will refer to this topology. A map $\phi : \mathcal{P}(X) \to [0,\infty]$ is a *submeasure on* X if $\phi(\emptyset) = 0$ and $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$, for all $A, B \subset X$. It is *lower semicontinuous* if additionally $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{x_0, \ldots, x_n\})$, where $X = \{x_0, x_1, \ldots\}$ is an enumeration of the set X. Mazur proved in [23] that $\mathcal{I} \in \mathbf{\Sigma^0_2}$ if and only if $\mathcal{I} = \mathbf{Fin}(\phi) = \{A \subset X : \phi(A) < \infty\}$ for some lower semicontinuous submeasure ϕ.

Notice that the submeasure ϕ on $\omega \times \omega$ given by

$$\phi(A) = \inf\bigl\{|\mathcal{C}| : A \subset \bigcup \mathcal{C} \text{ and each } C \in \mathcal{C} \text{ is either a generator of the first or of the second type of the ideal } \mathcal{WR}\bigr\}$$

is lower semicontinuous and $\mathcal{WR} = \mathbf{Fin}(\phi)$. Hence $\mathcal{WR}$ is $\mathbf{\Sigma^0_2}$.

We prove that $\mathcal{WR}$ is a critical ideal for weak Ramseyness. To define the latter notion we need some additional notation. If $s \in \omega^{<\omega}$, i.e., $s = (s(0), \ldots, s(k))$ is a finite sequence of natural numbers, then by $\mathrm{lh}(s)$ we denote its *length*, i.e., $k+1$. If $s, t \in \omega^{<\omega}$ and $\mathrm{lh}(s) \leq \mathrm{lh}(t)$, then we write $s \preceq t$ if $s(i) = t(i)$ for all