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Non-uniform spline recovery from small degree polynomial approximation

Yohann De Castro *, Guillaume Mijoule

Département de Mathématiques (CNRS UMR 8628), Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud 11, F-91405 Orsay Cedex, France

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ABSTRACT

We investigate the sparse spikes deconvolution problem onto spaces of algebraic polynomials. Our framework encompasses the measure reconstruction problem from a combination of noiseless and noisy moment measurements. We study a TV-norm regularization procedure to localize the support and estimate the weights of a target discrete measure in this frame. Furthermore, we derive quantitative bounds on the support recovery and the amplitude errors under a Chebyshev-type minimal separation condition on its support. Incidentally, we study the localization of the knots of non-uniform splines when a Gaussian perturbation of their inner-products with a known polynomial basis is observed (i.e. a small degree polynomial approximation is known) and the boundary conditions are known. We prove that the knots can be recovered in a grid-free manner using semidefinite programming.

1. Introduction

1.1. Non-uniform spline recovery

Our framework involves the recovery of non-uniform splines, i.e. a smooth polynomial function that is piecewise-defined on subintervals of different lengths. More precisely, we investigate a grid-free procedure to estimate a non-uniform spline from a polynomial approximation of small degree. Our estimation procedure can be used as a post-processing technique in various fields such as data assimilation [16], shape optimization [15] or spectral methods in PDE's [14].

For instance, one gets a polynomial approximation of the solution of a PDE when using spectral methods such as the Galerkin method. In this setting, one seeks a weak solution of a PDE using bounded degree polynomials as test functions. Then, the Lax–Milgram theorem grants the existence of a unique weak solution

* Corresponding author.
E-mail addresses: yohann.decastro@math.u-psud.fr (Y. De Castro), guillaume.mijoule@math.u-psud.fr (G. Mijoule).
URL: http://www.math.u-psud.fr/~decastro (Y. De Castro).

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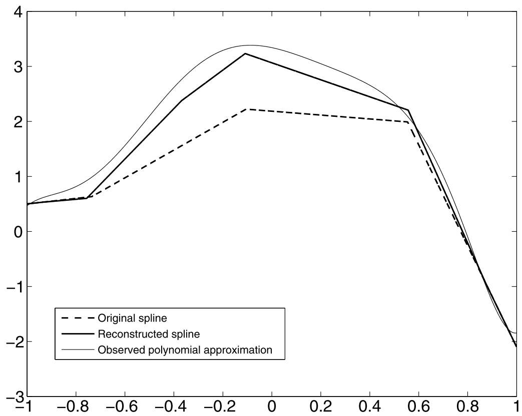

Fig. 1. Estimated spline (thick black line) of a non-uniform spline (thick dashed gray line) and its knots from a polynomial approximation (thin black line).

$\mathbf{f}$ for which a polynomial approximation P can be computed. Moreover, Céa's lemma shows that the Galerkin approximation P is comparable to the best polynomial approximation $\mathbf{p}(\mathbf{f})$ of the weak solution $\mathbf{f}$. This situation can be depicted by Assumption 1. Hence, if one knows the weak solution $\mathbf{f}$ is a non-uniform spline then our (post-processing) procedure can provide a grid-free estimate $\mathbf{f}$ from the Galerkin approximation P. Moreover, Theorem 2 shows that the recovered spline has large discontinuities near the large discontinuities of the target spline $\mathbf{f}$. Hence, the location of the large enough discontinuities of the weak solution $\mathbf{f}$ can be quantitatively and in a grid-free manner estimated from the Galerkin approximation using our algorithm.

As an example, Fig. 1 illustrates how our procedure improves a polynomial approximation of a non-uniform spline. Observe that discontinuities of splines make them difficult to approximate by polynomials. Consider an approximation (thin black line) of the spline (thick dashed gray line). It seems rather difficult to localize the discontinuities of the spline from the knowledge of this polynomial approximation and the boundary conditions. Nevertheless, our procedure produces a non-uniform spline (thick black line) whose large discontinuities are close to the knots of the target spline.

The method we propose is as follows. Following an idea of [3], we aim at reconstructing a spline of degree d by recovering its $d+1$ distributional derivative, using tools of the super-resolution theory [10,8,9]. More precisely, consider a univariate spline $\mathbf{f}$ of degree d, defined on $[-1,1]$. The $d+1$ distributional derivative of $\mathbf{f}$, denoted $\mathbf{f}^{(d+1)}$, is a discrete signed measure whose support are the knots of the spline. Using an integration by parts, one can show that the first $m+1$ polynomial moments of $\mathbf{f}^{(d+1)}$ can be expressed as a linear combination of the first $m-d$ moments of $\mathbf{f}$ and its $2(d+1)$ boundary conditions; moreover, the first $d+1$ moments of $\mathbf{f}^{(d+1)}$ only depend on the boundary conditions (details are in Lemma 5). As a consequence, observing $m-d$ noisy moments and the (noiseless) boundary conditions of $\mathbf{f}$ is equivalent to observing $d+1$ noiseless and $m-d$ noisy moments of $\mathbf{f}^{(d+1)}$. This observation is the motivation of the theoretical work of this paper.

1.2. *Sparse spikes deconvolution onto spaces of algebraic polynomials*

In this paper, we extend some recent results in spike deconvolution to the frame of algebraic polynomials. Beyond the theoretical interest, we focus on this model in order to bring tools and quantitative guarantees from the super-resolution theory to the companion problem of the recovery of knots of non-uniform splines [3]. At first glance, this setting can be depicted as a deconvolution problem where one wants to recover the location of the support of a discrete measure from the observation of its convolution with an algebraic polynomial of given degree m. More precisely, we aim at recovering a discrete measure from the knowledge of the true $(d+1)$ first moments and a noisy version of the $(m-d)$ next ones.