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Nontrivial solutions of discrete nonlinear equations with variable exponent

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ABSTRACT

In the present paper, we show the existence of ground state solution of a discrete $p(n)$-Laplacian type equation involving unbounded potential by using the Mountain-Pass theorem and Nehari manifold.

1. Introduction

We study the following difference equation

$$-\nabla^{-}(\left|\nabla^{+}u_n\right|^{p_n-2}\nabla^{+}u_n)+V_n\left|u_n\right|^{q_n-2}u_n=f_n\left(u_n\right),\ n\in\mathbb{Z} \tag{1.1}$$

on the integer lattice, where $\nabla^{+}u_n=u_{n+1}-u_n$ and $\nabla^{-}u_n=u_n-u_{n-1}$ are the forward and backward difference operators, respectively; V_n, $n\in\mathbb{Z}$, is a sequence of real numbers, and $f_n\left(t\right):\mathbb{R}\to\mathbb{R}$, $n\in\mathbb{Z}$, is a continuous function. We impose the following boundary condition at infinity

$$\lim_{|n|\to\infty}u_n=0\,, \tag{1.2}$$

i.e., we are looking for homoclinic solutions.

Equation (1.1) is the discrete counterpart of the following nonlinear differential equation

$$-(\left|u'\right|^{p(x)-2}u')'+V\left(x\right)\left|u\right|^{q(x)-2}u_n=f(x,u) \tag{1.3}$$

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Equations of the form (1.3), as well as their multi-dimensional versions, appear in many applications, such as fluid dynamics and nonlinear elasticity, to name a few (see, *e.g.*, [2,18] and references therein). In the case when $p(x) = q(x) = 2$, (1.3) becomes the stationary nonlinear Schrödinger equation (NLS). It has an enormous number of applications, for instance, in nonlinear optics [10] and condensed matter physics [8].

As in the case of equation (1.3), equation (1.1) reduces to the stationary discrete nonlinear Schrödinger equation (DNLS) when $p_n = q_n = 2$. As its continuous counterpart, DNLS has many applications in various areas of physics (see, *e.g.*, [3,4,7]). On the other hand, there is a number of rigorous results about this equation. Here we only mention papers [13,14,20,21] in which the existence of solutions satisfying (1.2) is studied by means of variational techniques. In contrast, to the best of our knowledge [9] is the only paper dealing with general problem (1.1), (1.2). The main result of that paper concerns the existence of nontrivial solutions under the assumption that the sequences p_n, V_n and $f_n(\cdot)$ are periodic in n, with the same period. Also we point out that in [5] the existence of a nontrivial solution is obtained under the assumption that p_n is independent of n while the potential V_n is infinitely growing at infinity. The Dirichlet problem for discrete p-Laplacian and $p(n)$-Laplacian equations are studied in [11] and [12] respectively (see also [17]).

We will study problem (1.1), (1.2) under the following assumptions. For any sequence r_n, $n \in \mathbb{Z}$, of real numbers we set

$$r^- = \inf_{n\in\mathbb{Z}} r_n \quad \text{and} \quad r^+ = \sup_{n\in\mathbb{Z}} r_n \,.$$

Throughout the paper we always assume that $1 < p^- \le p^+ < \infty$, $1 < q^- \le q^+ < \infty$ and $q^+ \le p^-$.

(V1) The potential sequence V_n is such that $V_n \ge \alpha_0 > 0$ for all $n \in \mathbb{Z}$, and $V_n \to +\infty$ as $|n| \to \infty$.

(f1) The function $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{Z}$, is continuous. Moreover, for every $R > 0$ there exists a positive constant $C(R)$ such that

$$|f_n(t)| \le C(R) \;\; \forall (n,t) \in \mathbb{Z}\times\mathbb{R}, \;\; |t| \le R \,.$$

(f2) There exists $\theta > p^+$ such that

$$0 < \theta F_n(t) := \theta \int_0^t f_n(s)\,ds \le f_n(t)\,t \;\forall n \in \mathbb{Z}, t \in \mathbb{R}\setminus\{0\} \;.$$

(f3) $f_n(t) = o\left(|t|^{q^+-1}\right)$ as $t \to 0$ uniformly in $n \in \mathbb{Z}$.

(f4) $\frac{f_n(t)}{|t|^{q^+-1}}$ is an increasing function of t on $\mathbb{R}\setminus\{0\}$ for every $n \in \mathbb{Z}$.

(f5) $f_n(-t) = -f_n(t)$.

Remark 1.1. The function $g_n(t) = |t|^{\sigma_n-2}\,t$, where $\sigma^- > q^+$, satisfies assumptions (f1)–(f5).

Remark 1.2. Assumptions (f2) and (f3) imply that for any $\varepsilon > 0$ there exists a bounded sequence $c_n = c_n(\varepsilon) > 0$ such that

$$F_n(t) \ge -\varepsilon|t|^{q^+} + c_n|t|^\theta$$

for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.