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Operator space and operator system analogs of Kirchberg's nuclear embedding theorem

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ABSTRACT

The Gurarij operator space $\mathbb{NG}$ introduced by Oikhberg is the unique separable 1-exact operator space that is approximately injective in the category of 1-exact operator spaces and completely isometric linear maps. We prove that a separable operator space X is nuclear if and only if there exist a linear complete isometry $\varphi : X \to \mathbb{NG}$ and a completely contractive projection from $\mathbb{NG}$ onto the range of φ. This can be seen as the operator space analog of Kirchberg's nuclear embedding theorem. With similar methods we also establish the natural operator system analog of Kirchberg's nuclear embedding theorem involving the Gurarij operator system $\mathbb{GS}$.

1. Introduction

Nuclearity and exactness are properties of fundamental importance for the theory of C*-algebras and the classification program. The celebrated Kirchberg exact embedding theorem characterizes (up to *-isomorphism) the separable exact C*-algebras as the C*-subalgebras of the Cuntz algebra $\mathcal{O}_2$ [26, Theorem 6.3.11]; see also [14,15]. Furthermore the Kirchberg nuclear embedding theorem asserts that (up to *-isomorphism) the separable nuclear C*-algebras are precisely the C*-subalgebras of $\mathcal{O}_2$ that are the range of a (completely) contractive projection [26, Theorem 6.3.12]. The Cuntz algebra $\mathcal{O}_2$, initially introduced and studied in [6], is the C*-algebra generated by two isometries of the Hilbert space with orthogonally complementary ranges.

The main result of this paper is an analog of Kirchberg's nuclear embedding theorem for operator spaces involving the *(noncommutative) Gurarij operator space*. This is the unique separable 1-exact operator space $\mathbb{NG}$ which is approximately injective in the category of 1-exact operator spaces with completely isometric

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maps. In other words $\mathbb{NG}$ is characterized by the following property: for any $n \in \mathbb{N}$, $\varepsilon > 0$, finite-dimensional 1-exact operator spaces $E \subset F$, and complete isometry $\phi : E \to \mathbb{NG}$, there exists a complete isometry $\psi : F \to \mathbb{NG}$ such that $\left\|\psi_{|E} - \phi\right\| \leq \varepsilon$. Such a space can be thought as a noncommutative analog of the Gurarij Banach space $\mathbb{G}$, which is defined by the same property as above where one considers Banach spaces instead of 1-exact operator spaces [11,21,17].

Existence of the Gurarij operator space was first proved in [22], while uniqueness was established in [19]. It is worth mentioning here that, albeit being 1-exact, $\mathbb{NG}$ does not admit any completely isometric embedding into an exact C*-algebra by [20, Corollary 4.16]. In particular $\mathbb{NG}$ is not completely isometric to a C*-algebra.

A separable operator space is 1-exact if and only if it admits a completely isometric embedding into $\mathbb{NG}$ [19, Theorem 4.3]. (Modulo uniqueness of $\mathbb{NG}$, this also follows from [22, Theorem 1.1] and the proof of [8, Theorem 4.7].) Such a result can be regarded as an operator space version of Kirchberg's exact embedding theorem. In this paper we prove the natural analog of Kirchberg's nuclear embedding theorem for operator spaces: the separable nuclear operator spaces are (up to completely isometric isomorphism) precisely the subspaces of $\mathbb{NG}$ that are the range of a completely contractive projection; see Theorem 2.5.

We also observe that all the results mentioned above hold for operator systems, as long as one only considers *unital* linear maps, and replaces the Gurarij operator space $\mathbb{NG}$ with the Gurarij operator system $\mathbb{GS}$. Such an operator system can be characterized similarly as $\mathbb{NG}$, where one considers complete order embeddings instead of complete isometries. Existence and uniqueness of the Gurarij operator system, as well as universality among separable 1-exact operator systems, have been established in [20]. The analog of Kirchberg's nuclear embedding theorem in this context asserts that the separable nuclear operator systems are (up to complete order isomorphism) precisely the subsystems of $\mathbb{GS}$ that are the range of a unital completely positive projection; see Theorem 3.3.

It is worth mentioning that it should come as no surprise that, while the Kirchberg embedding theorems involve the Cuntz algebra $\mathcal{O}_2$, their operator space and operator system versions involve the Gurarij operator space and operator system. In fact by [15, Theorem 1.13 and Theorem 2.8] any separable exact C*-algebra embed into $\mathcal{O}_2$, and any two unital embeddings of a separable unital exact C*-algebra into $\mathcal{O}_2$ are approximately unitarily equivalent. It therefore follows from [1, Theorem 2.21] that finitely generated exact unital C*-algebras form a Fraïssé class with limit $\mathcal{O}_2$. The Gurarij operator space and operator systems can be similarly described as the Fraïssé limits in the sense of [1] of the classes of finitely-generated 1-exact operator spaces and operator systems; see [19,20]. It is therefore natural to expect that they exhibit similar properties as $\mathcal{O}_2$ in the respective categories.

The present paper is organized as follows. In Section 2 we prove the above mentioned results concerning operator spaces. We recall a few basic notions about operator spaces in Subsection 2.1. Subsection 2.2 contains canonical approximate amalgamation results for operator spaces, while a characterization of nuclearity is recalled in Subsection 2.3. The operator space analog of Kirchberg's nuclear embedding theorem is then proved in Subsection 2.4. The analogous results for operator systems are stated in Section 3. The proofs are similar, and only the relevant changes to be made from the operator space case will be pointed out.

2. Nuclear operator spaces

2.1. Some background notions on operator spaces

An *operator space* X is a closed subspace of the algebra $B(H)$ of bounded linear operators on a complex Hilbert space H. The identification of the space $M_n(X)$ of $n \times n$ matrices with entries in X with a space of operators on the n-fold Hilbertian sum $H^{\oplus n}$ induces, for every $n \in \mathbb{N}$, a norm $M_n(X)$ (the operator norm). The *matricially normed* vector spaces that arise in this way have been characterized by Ruan [27, Theorem 3.1] as those satisfying the following condition: