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The density of the solution to the stochastic transport equation with fractional noise

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A B S T R A C T

We consider the transport equation driven by the fractional Brownian motion. We study the existence and the uniqueness of the weak solution and, by using the tools of the Malliavin calculus, we prove the existence of the density of the solution and we give Gaussian estimates from above and from below for this density.

1. Introduction

The purpose of this paper is to study the probability law of the real-valued solution of the following stochastic partial differential equations

$$\begin{cases} du(t,x) + b(t,x)\nabla u(t,x)\ dt + \nabla u(t,x) \circ dB_t^H + F(t,u)\ dt = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $B_t^H = (B_t^{H_1}, \dots, B_t^{H_d})$ is a fractional Brownian motion (fBm) in $\mathbb{R}^d$ with Hurst parameter $H = (H_1, \dots, H_d) \in \left[\frac{1}{2}, 1\right)^d$ and the stochastic integration is understood in the symmetric (Stratonovich) sense. The equation (1) is usually called the stochastic transport equation and arises as a prototype model in

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a wide variety of phenomena. Although we introduced (1) in a general form, we mention that some results will be obtained in dimension one.

The stochastic transport equation with standard Brownian noise has been first studied in the celebrated works by Kunita [11,12] and more recently it has been the object of study for many authors. We refer, among many others, to [4,7,8,14,15,19].

Our aim is to analyze the stochastic partial equation (1) when the driving noise is the fractional Brownian motion, including the particular case of the Brownian motion. We will first give, by interpreting the stochastic integral in (1) as a symmetric integral via regularization in the Russo–Vallois sense [21], an existence and uniqueness result for the weak solution to (1) via the so-called method of characteristics and we express the solution as the initial value applied to the inverse flow generated by the equation of characteristics. This holds, when $H_i = \frac{1}{2}$, $i = 1, .., d$ for any dimension d and in dimension $d = 1$ if the Hurst parameter is bigger than one half. Using this representation of the solution to (1), we study the existence and the Gaussian estimates for its density via the analysis of the dynamic of the inverse flow. The classical tools to study the absolute continuity of the law of random variables with respect to the Lebesgue measure is the Malliavin calculus. We refer to the monographs [17] or [23] for various applications of the Malliavin calculus to the existence and smoothness of the density of random variables in general, and of solutions to stochastic equations in particular.

We will prove the Malliavin differentiability of the solution to (7) by analyzing the dynamics of the inverse flow generated by the characteristics (9). Using a result in [16] we obtain, in dimension $d = 1$ upper and lower Gaussian bounds for the density of the solution to the transport equation. We are also able to find the explicit form of the density in dimension $d \geq 2$ when the driving noise is the standard Brownian motion and the drift is divergence-free (i.e. the divergence of the drift vanishes).

We organized our paper as follows. In Section 2 we recall the existence and uniqueness results for the solution to the transport equation driven by the standard Brownian motion. In Section 3, we analyze the weak solution to the transport equation when the noise is the fBm, via the method of characteristics. In Section 4 we study the Malliavin differentiability of the solution to the equation of characteristics and this will be applied in Section 4 to obtain the existence and the Gaussian estimates for the solution to the transport equation. In Section 6 we obtain an explicit formula for the density when the noise is the Wiener process and the drift is divergence-free.

2. Stochastic transport equation driven by standard Brownian motion

Throughout the paper, we will fix a probability space $(\Omega, \mathcal{F}, P)$ and a d-dimensional Wiener process $(B_t)_{t\in[0,T]}$ on this probability space. We will denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by B.

We will start by recalling some known facts on the solution to the transport equation driven by a standard Wiener process in $\mathbb{R}^d$.

The equation (1) is interpreted in the strong sense, as the following stochastic integral equation

$$u(t,x) = u_0(x) - \int_0^t b(s,x)\nabla u(s,x)\, ds - \sum_{i=0}^{d} \int_0^t \partial_{x_i} u(s,x) \circ dB_s^i - \int_0^t F(t,u)\, ds \qquad (2)$$

for $t \in [0,T]$ and $x \in \mathbb{R}^d$.

The solution to (1) is related with the so-called equation of characteristics. That is, for $0 \leq s \leq t$ and $x \in \mathbb{R}^d$, consider the following stochastic differential equation in $\mathbb{R}^d$

$$X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x))\, dr + B_t - B_s, \qquad (3)$$

and denote by $X_t(x) := X_{0,t}(x), t \in [0,T], x \in \mathbb{R}^d$.