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Thin viscous films: Thinning driven by surface-tension energy dissipation

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ABSTRACT

We study the evolution of a thin film of fluid modeled by the lubrication approximation for thin viscous films. We prove existence of (dissipative) strong solutions for the Cauchy problem when the sub-diffusive exponent ranges between 3/8 and 2; then we show that these solutions tend to zero at rates matching the decay of the source-type self-similar solutions with zero contact angle. Finally, we introduce the weaker concept of dissipative mild solutions and we show that in this case the surface-tension energy dissipation is the mechanism responsible for the H^1-norm decay to zero of the thickness of the film at an explicit rate. Relaxed problems, with second-order nonlinear terms of porous media type are also successfully treated by the same means.

1. Introduction

The equation analyzed in this paper is

$$\begin{cases} u_t = -(u^n u_{xxx})_x & \text{in } Q := \mathbb{R} \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{1.1}$$

Equation (1.1) is derived from a lubrication approximation and models the surface-tension dominated motion of thin viscous films and spreading droplets [8]. The goal pursued in this paper is two-fold. First, we prove existence of dissipative strong solutions for the Cauchy problem, thus extending the result by Carrillo and Toscani [7] ($n = 1$) to $3/8 < n < 2$. As in [7], these solutions arise as limits of the compactly supported strong solutions found in [1]. Consequently, they will satisfy the same rates of decay as the ones proved in [1] to be optimal (they match the decay rates of the self-similar solutions with zero contact angle). Secondly, we consider *dissipative* weak solutions with slightly higher regularity, which we call *dissipative mild solutions*, for the Cauchy problem and show that they asymptotically converge to zero at an explicit rate. The point we

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make is that the surface-tension energy dissipation is the mechanism responsible for the asymptotic decay. The model itself is derived from the Navier–Stokes equation when the typical thickness of the film is much smaller than the typical length of the region occupied by the fluid [11]. Furthermore, nonnegative solutions u satisfying, for some $t_0 \geq 0$, $u(\cdot, t_0) \in H^2(\mathbb{R})$ have the obvious property that $u_x(x_0, t_0) = 0$ whenever $u(x_0, t_0) = 0$. This is known as the zero contact angle property. It is expected for a physically correct model that the thickness of the film decays to zero as $t \to \infty$. Therefore, we argue that *physical* solutions should be *dissipative*, i.e. they should satisfy an inequality of the form (2.6).

A qualitative comparison between (1.1) and second-order degenerate parabolic problems of porous media type can be found, for example, in [5]. In essence, the common features exhibited are the parabolicity, the divergence structure and the existence of nonnegative solutions corresponding to nonnegative initial data. The main difference is the lack of a maximum principle for (1.1). As a consequence, (1.1) is analyzed via dissipation of certain nonlinear entropies.

While our result has no bearing on the important question of decay to self-similarity in the L^1-norm, it does generalize the findings of Carlen and Ulusoy [6] in two different directions. First, we carry out our analysis for the case of dissipative mild solutions, while the analysis in [6] is only formal (as existence of classical solutions is still open). Secondly, we deal with an entire array of PDE's of the type (1.1) for $0 < n < 2$, as opposed to [6] where the case $n = 1$ alone is analyzed.

Furthermore, convergence in the sup-norm for the strong solution to the self-similar solution was proved in [7] for $n = 1$. The rate obtained in [7] was $(t + 1)^{-1/15}$. Here we show that this convergence holds for weak solutions with zero contact angle as well and is simply a consequence of the uniform decay to zero; we also improve the rate to $(t + 1)^{-1/8}$ while at the same time we show that it holds true for all $0 < n < 2$. Of course, the main purpose of [7] was proving decay of strong solutions to the self-similar profiles which known to exist for $0 < n < 3$. The important aspect here is that the only relevant norm for this decay is the L^1-norm. Indeed, we show in this paper that convergence to self-similarity in any L^p for $1 < p \leq \infty$ is merely a manifestation of the decay to zero of the film thickness.

The existence of nonnegative weak solutions for this problem in a bounded domain was proved by Bernis and Friedman [2] for $n \geq 1$, and Bertozzi and Pugh [5] for all $n > 0$. In [1], the author shows that strong solutions for the natural boundary-value problem (first and third derivatives are zero at the boundary) can be used to construct a strong solution for the Cauchy problem. These solutions are obtained as accumulation points in the appropriate topology of classical solutions for regularized problems of the type

$$\begin{cases} u_{\varepsilon,t} = -(f_\varepsilon(u_\varepsilon)u_{\varepsilon,xxx})_x & \text{in } (-a, a) \times (0, \infty), \\ u_\varepsilon(\cdot, 0) = u_{0\varepsilon} & \text{in } (-a, a) \end{cases} \tag{1.2}$$

with boundary conditions

$$u_{\varepsilon,x}(\pm a, t) = 0 = u_{\varepsilon,xxx}(\pm a, t) \text{ for } t > 0,$$

where $f_\epsilon(s)$ is an appropriate one-parameter modification of the function s^n with the purpose of avoiding parabolic degeneracy [2]. Uniqueness of these strong solutions is not known. Thus, the results in [1] and [7] are proved for the strong solutions arising as limits of these standard regularized problems. The rates of uniform decay to zero obtained in [1] for $0 < n < 2$ are shown to be optimal, in the sense that they match the rates of decay for the corresponding self-similar solutions [3]

$$h(x, t) = t^{-\alpha} H\big(t^{-\alpha} x\big), \ \alpha = \frac{1}{n + 4},$$

where H is even, compactly supported, solves $H^n(x)H'''(x) = \alpha x H(x)$ and satisfies $H' = 0$ at the edge of the support.