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The Lerch transcendent from the point of view of Fourier analysis ☆

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ABSTRACT

We obtain some well-known expansions for the Lerch transcendent and the Hurwitz zeta function using elementary Fourier analytic methods. These Fourier series can be used to analytically continue the functions and prove the classical functional equations, which arise from the relations satisfied by the Fourier conjugate and flat Fourier series. In particular, the functional equation for the Riemann zeta function can be obtained in this way without contour integrals. The conjugate series for special values of the parameters yields analogous results for the Bernoulli and Apostol–Bernoulli polynomials. Finally, we give some consequences derived from the Fourier series.

1. Introduction

Recall that the Lerch transcendent function is defined via the series

$$\Phi(\lambda, s, z) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+z)^s}$$

for complex parameters λ, s and z. The closely related Lerch zeta function, where the first parameter a is exponential, is

$$L(a, s, z) = \sum_{k=0}^{\infty} \frac{e^{2\pi i k a}}{(k+z)^s} \tag{1.1}$$

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(logarithms and powers are always taken to be principal). These functions, introduced by M. Lerch in [7], are of great interest because their analytic continuations include as special cases important transcendental functions such as the polylogarithm family, the Hurwitz zeta function and of course the Riemann zeta function.

When dealing with the Lerch functions, for example in analytic number theory, one often encounters the Fourier series

$$F(a,s,x)=\sum_{n\in\mathbb{Z}}\frac{e^{2\pi inx}}{(n+a)^s}, \tag{1.2}$$

where $\mathbb{Z}$ denotes the set of integers. The relationships between (1.1) and Fourier series like (1.2) are well known and usually proved via complex analytic methods involving contour integrals. In previous papers (see [11,12]), we have derived some properties of the Lerch function based on Fourier series by direct calculation of the Fourier coefficients. Observe, for instance, the simplicity of the following reasoning. Let $\operatorname{Im} a<0$ and $\operatorname{Re} s<0$; then, for each $k\in\mathbb{Z}$ we have

$$\begin{aligned}\int_0^1 e^{-2\pi iax}L(-a,x,s)e^{-2\pi ikx}\,dx &= \sum_{n=0}^{\infty}\int_0^1 \frac{e^{-2\pi ia(x+n)}}{(x+n)^s}\,e^{-2\pi ikx}\,dx\\ &= \int_0^{\infty} e^{-2\pi i(k+a)t}t^{-s}\,dt=\Gamma(1-s)(2\pi i(k+a))^{s-1}\end{aligned}$$

(the exchange of series and integrals is easily justifiable under the given restrictions on the parameters). Consequently, by Dirichlet's theorem,

$$L(-a,s,x)=e^{2\pi iax}\Gamma(1-s)\sum_{n\in\mathbb{Z}}\frac{e^{2\pi inx}}{(2\pi i(n+a))^{1-s}}. \tag{1.3}$$

The series on the right-hand side is not exactly a series of the type (1.2) since, in general, the parameter s is complex and the term $(2\pi i)^{1-s}$ is not a common factor. In fact, we will see that the series in (1.3) is related to the *conjugate* series of the Fourier series (1.2). The Lerch zeta function (1.1) will be a *flat series*, namely, the projection on the positive coefficients of a Fourier series of the type (1.2).

In this paper, we address in a systematic way the study of these conjugate and flat functions. For example, (c) in Theorem 7 establishes a new functional relation, between the Lerch zeta function and the conjugate of the function defined by the series $F(a,s,x)=\sum_{n\in\mathbb{Z}}(n+a)^{-s}e^{2\pi inx}$, whose flat function is the periodic Hurwitz zeta function. Analogously, Theorem 11 relates the Hurwitz zeta function to its flat and conjugate functions.

We show that it is possible to obtain the functional equation of the Lerch transcendent (Theorem 10) with these real-analytic tools. For $a=0$, corresponding to the Hurwitz zeta function, the calculation of the Fourier series by real techniques is somewhat less straightforward, but also possible. The study of conjugate series in this case allows us to prove the functional equation for the Riemann zeta function (Corollary 12) with real methods. One advantage of this approach is that it reveals that many well-known classical results have conjugate counterparts. Another proof of the functional equation for the Riemann zeta function via a different real method has been presented in [5, Section 6.2].

Some problems arise for certain values of the parameters corresponding to the Apostol–Bernoulli and Bernoulli polynomials. We introduce the conjugate series of the Apostol–Bernoulli polynomials, along the lines initiated in [6] for Bernoulli polynomials, obtaining new results about their Fourier series (5.4), generating function, and various new analogous relations, such as a Möbius inversion formula for the conjugate Bernoulli polynomials (Theorem 17).