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On functional equations of the multiplicative type

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ABSTRACT

This paper aims to determine the general solution $f : \mathbb{F}^2 \to S$ of the equation $f(\phi(x, y, u, v)) = f(x, y)\, f(u, v)$ for suitable conditions on the function $\phi : \mathbb{F}^4 \to \mathbb{F}^2$, where $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$, and S is a multiplicative semigroup. Using this result, we determine the general solution of several functional equations studied earlier, namely $f(ux+vy,\ uy+vx) = f(x, y)\, f(u, v)$; $f(ux+(\lambda-1)vy,\ vx+uy+(\lambda-2)vy) = f(x, y)\, f(u, v)$; $f(ux-vy,\ uy-vx) = f(x, y)\, f(u, v)$; $f(ux+vy,\ uy-vx) = f(x, y)\, f(u, v)$; $f(ux+vy,\ uy-vx) = f(x, y)\, f(v, u)$; and $f(ux-vy,\ uy+v(x+y)) = f(x, y)\, f(u, v)$.

1. Introduction

In [3], Blecksmith and Broudno gave a simple proof of the following result: *For every positive integer m, there is an integer n with at least m proper representations as the sum of three fourth powers.* If one lets the quadratic form on the right hand side of the *Proth identity*

$$x^4 + y^4 + (x+y)^4 = 2\left(x^2 + xy + y^2\right)^2 \tag{1.1}$$

be

$$f(x, y) = x^2 + xy + y^2 \tag{1.2}$$

then it can be easily verified that

$$f(ux - vy, uy + v(x+y)) = f(x, y)\, f(u, v) \tag{1.3}$$

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for all $x, y, u, v \in \mathbb{R}$. It is obvious that f given in (1.2) is a solution of the above functional equation (1.3). To prove the above mentioned result, Blecksmith and Broudno [3] used the Proth identity and the above functional equation (1.3). However, the general solution of the above equation was not given in [3].

The question that arises is whether $f(x, y) = x^2 + xy + y^2$ is the only solution of (1.3) or there are other solutions. This question was addressed by Chávez and Sahoo [4] (and subsequently fixed an error in [4] by Chung and Sahoo [5]) in the following theorem.

Theorem 1.1. *The general solution $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation*

$$f(ux - vy, uy + v(x + y)) = f(x, y)\, f(u, v) \tag{1.4}$$

for all $u, v, x, y \in \mathbb{R}$ is given by $f \equiv 1$ or

$$f(x, y) = M(x^2 + xy + y^2)\, e^{A\left(\arctan\left(\frac{\sqrt{3}y}{2x+y}\right)\right)}, \quad f(0, 0) = 0 \tag{1.5}$$

for all $x, y \in \mathbb{R} \setminus \{0\}$, where $M : (0, \infty) \to \mathbb{R}$ is a multiplicative function and $A : \mathbb{R} \to \mathbb{R}$ is an additive function satisfying $A(2\pi) = 0$.

One of the well known properties of matrices is the following: The determinant of the product of two square matrices is the product of their determinants. If

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u & v \\ v & u \end{pmatrix}$$

are any two symmetric matrices, then

$$det\left(\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} u & v \\ v & u \end{pmatrix}\right) = det\begin{pmatrix} x & y \\ y & x \end{pmatrix}\, det\begin{pmatrix} u & v \\ v & u \end{pmatrix},$$

which is

$$det\begin{pmatrix} ux + vy & uy + vx \\ uy + vx & ux + vy \end{pmatrix} = det\begin{pmatrix} x & y \\ y & x \end{pmatrix}\, det\begin{pmatrix} u & v \\ v & u \end{pmatrix}. \tag{1.6}$$

If we define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = det\begin{pmatrix} x & y \\ y & x \end{pmatrix} \tag{1.7}$$

for all $x, y \in \mathbb{R}$, then we arrive at the functional equation

$$f(ux + vy,\, uy + vx) = f(x, y)\, f(u, v) \tag{1.8}$$

for all $x, y, u, v \in \mathbb{R}$. Since

$$det\begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 - y^2,$$

it is clear that $f(x, y) = x^2 - y^2$ is a solution of (1.8). The question that arises is whether $f(x, y) = x^2 - y^2$ is the only solution or there are other solutions. Chung and Sahoo in [6] have shown that there are other solutions besides $f(x, y) = x^2 - y^2$. Chung and Sahoo in [6] proved the following theorem: