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Topology on new sequence spaces defined with wavelet leaders

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Using wavelet leaders instead of wavelet coefficients, new sequence spaces of type $\mathcal{S}^\nu$ are defined and endowed with a natural topology. Some classical topological properties are then studied; in particular, a generic result about the asymptotic repartition of the wavelet leaders of a sequence in $\mathcal{L}^\nu$ is obtained. Eventually, comparisons and links with Oscillation spaces are also presented as well as with $\mathcal{S}^\nu$ spaces.

1. Introduction

The study of the regularity of a signal by means of its wavelet coefficients is now a widely used tool. Mathematically, it involves the use of sequence spaces which are supposed to constitute an appropriate setting to handle the information. In order to study the regularity of a signal via the distribution of its wavelet coefficients, $\mathcal{S}^\nu$ spaces have been introduced and it has been shown that they contain more information than the classical Besov spaces (see [13]). Nevertheless, the use of these $\mathcal{S}^\nu$ spaces presents some weaknesses and then, new spaces of the same type have recently been introduced using wavelet leaders instead of wavelet coefficients (see [8]). These spaces are denoted by $\mathcal{L}^\nu$.

Before giving more details about the introduction and the definition of $\mathcal{L}^\nu$ spaces, let us be more precise about the notion of regularity. Let $x_0 \in \mathbb{R}$ and $\alpha \geq 0$. A locally bounded function $f : \mathbb{R} \to \mathbb{R}$ belongs to the Hölder space $C^\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P of degree strictly less than α such that

$$|f(x) - P(x)| \leq C|x - x_0|^\alpha$$

for all x in a neighbourhood of x_0. The Hölder exponent of f at x_0 is defined by

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$$h_f(x_0) := \sup\{\alpha \geq 0 : f \in C^\alpha(x_0)\}$$

and the multifractal spectrum of f is the function d_f defined by

$$d_f(h) := \dim_{\mathcal{H}}\{x \in \mathbb{R} : h_f(x) = h\}, \quad h \in [0, +\infty]$$

(where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension). This function gives a geometrical idea about the distribution of the singularities of f. For a general signal (i.e. a function obtained from real-life data), it is clearly impossible to estimate d_f numerically since it involves the successive determination of several intricate limits. Therefore one tries instead to estimate this spectrum from quantities which are numerically computable. Such a method is called a multifractal formalism.

The Frisch–Parisi conjecture, classically used, gives such an estimation based on a wavelet decomposition and the use of Besov spaces (see [23,12]). Nevertheless, it appeared that this use of Besov spaces is not sufficient to handle all the information concerning the pointwise regularity contained in the distribution of the wavelet coefficients (see [13]). In particular, it can only lead to recover increasing and concave hull of spectra.

In order to get a suitable context to obtain multifractal results in the non-concave case, $\mathcal{S}^\nu$ spaces have then been introduced (see [13]). These spaces contain the maximal information that can be derived from the repartition at every scale of the wavelet coefficients of a function. They have been studied in several papers: topological (and specific functional analysis) results were obtained, as well as answers for multifractal formalisms (see [5,6,4,1,3,2]). An implementation of this formalism has been proposed and tested on several theoretical examples in [18]. However, the $\mathcal{S}^\nu$ spaces can only detect increasing part of spectra.

Meanwhile, it appeared that more accurate information concerning the pointwise regularity can be obtained when relying on wavelet leaders, which can be seen as local suprema of wavelet coefficients. Indeed, wavelet leaders give an easier characterization of the pointwise regularity than wavelet coefficients (see for example [16] and references therein). In particular, they allow to obtain information about the inter-scale organization of the wavelet coefficients, without making any a priori probabilistic assumptions on their repartition. In this context, Oscillation spaces have been introduced as a generalization of Besov spaces using wavelet leaders (see [15]) and multifractal results have been obtained (see [14,16]). In particular, Oscillation spaces gives a method which allows to recover increasing and decreasing parts of spectra. Nevertheless, this method is still limited to concave spectra. So, a natural idea was to extend the study of the $\mathcal{S}^\nu$ spaces (defined directly using the wavelet coefficients) to the context of wavelet leaders. Those spaces, called $\mathcal{L}^\nu$ spaces and introduced in [8], lead to better approximations for non-concave spectra with a decreasing part. Several positive results have been obtained in [8]. Moreover, in [18], the different formalisms (based on Oscillation spaces, $\mathcal{S}^\nu$ spaces and $\mathcal{L}^\nu$ spaces) have been compared. It appeared that the method based on the $\mathcal{L}^\nu$ spaces is more efficient from the theoretical point of view and that in practice, it gives complementary results to those obtained using the formalism based on Oscillation spaces.

In this paper, in order to understand better the structure of the $\mathcal{L}^\nu$ spaces, we endow them with a topology. As done in the case of the $\mathcal{S}^\nu$ spaces (see [3,4,6]), one of our purposes is to get applications in multifractal analysis and in particular, to obtain the generic validity of the multifractal formalism based on $\mathcal{L}^\nu$ spaces. This would give a theoretical justification to this method. Indeed, as for the other multifractal formalisms, the method based on the $\mathcal{L}^\nu$ spaces never holds in complete generality, but it yields an upper bound for the multifractal spectrum of all functions in the space $\mathcal{L}^\nu$ (see [8]). This is the best that can be expected: usually, there are no non-trivial minorations for the multifractal spectrum of all functions in the space. Nevertheless, one can hope that for most of the functions in the space, that is to say for a generic subset of the space (in the sense of Baire categories), the inequality becomes an equality.

Let us give some classical notations used in the paper. The set of strictly positive natural numbers is $\mathbb{N}$ and we denote $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. We use the notation λ to refer to the dyadic interval