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Extremal curves of the total curvature in homogeneous 3-spaces

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A B S T R A C T

The space of curves which are extremal for the total curvature energy is well understood in isotropic homogeneous 3-spaces, said otherwise, spaces of constant curvature. In this paper we obtain that space of extremals in homogeneous 3-spaces whose isometry group has dimension four, that is, rotationally symmetric homogeneous 3-spaces. Most of the geometry in these spaces is governed by the existence of a unit Killing vector field, ξ, sometimes called the Reeb vector field, which turns the homogeneous 3-space into the source of a Riemannian submersion whose target space is a surface with constant curvature. Here, we show that a curve is an extremal of the total curvature energy if and only if ξ lies into either the rectifying plane or the osculating plane along that curve. Then, we prove that every rotationally symmetric homogeneous 3-space, except $\mathbb{H}^2 \times \mathbb{R}$, admits a real one-parameter class of extremals with horizontal normal (Lancret helices). The whole family of extremals is completed with a second class made up of those curves with horizontal binormal. In contrast with the first class, it appears in any rotationally symmetric space, with no exception, and it can be modulated in the space of real valued functions. We also work out geometric algorithms to solve the so called solving natural equations for extremals problem, allowing us to determine them explicitly in many cases. In addition, we solve the closed curve problem by showing the existence of two families of closed extremals. Namely, a rational one-parameter class of closed Lancret helices that appears at any rotationally symmetric homogeneous 3-space, except in $\mathbb{H}^2 \times \mathbb{S}^1$, and a second class of extremals with horizontal binormal, which can be identified with the class of convex closed curves in the Euclidean plane. We also present a quantization principle, à la Dirac, for extremal values of the total curvature energy acting on closed curves in any rotationally symmetric homogeneous 3-space.

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1. Introduction

According to a classical result of Whitney and Grauestein, the total curvature of plane curves is, under suitable boundary conditions, an invariant in the homotopy class of the curve. Even more, each homotopy class of curves, satisfying suitable boundary conditions, is completely characterized by the common total curvature. As an obvious consequence, if fluctuations do not change the topology of curves, then a variational approach to the total curvature action of curves in the Euclidean plane turns out to be trivial.

On the other hand, another classical and well known result, due to Fenchel, assures that the total curvature of any simple closed curve in the Euclidean 3-space satisfies

$$\int_{\gamma} \kappa(s)\, ds \geq 2\pi,$$

with equality if and only if γ is a convex plane curve. Therefore, the minimum of the total curvature action over simple closed curves in the Euclidean space is 2π and it is reached just on the convex plane curves. Hence, an obvious and naive question is to decide whether, besides those minima, there is any other extremal (critical) of the total curvature functional acting on closed curves in the Euclidean space.

More precisely, it seems natural to consider the total curvature functional acting on a suitable space of curves in a certain surface, or more generally in a Riemannian space, and then to study the associated variational problem. Although the study of the total curvature of curves in Riemannian spaces has been intensively considered along the literature (see [8] and references therein), the systematic study of the associated variational approach was initiated in [1,2]. However, we have to point out that M.S. Plyushchay [13], proposed it as a model to study massless particles with rigidity.

Some progress has been made in this direction which, as far as we know, can be briefly summarized as follows. The variational approach for the total curvature in surfaces was first considered in [2], where it was shown that extremals of the total curvature are reached by curves consisting of parabolic points. Stability of extremals was also holographically characterized there. Then, the problem in spaces with the highest rigidity (constant curvature) was considered in [1], where it was shown that the dynamics associated with the total curvature action is consistent only in round 3-spheres. More precisely, when studying the total curvature dynamics of curves in an n-dimensional Riemannian space, $M^n(k)$, with constant curvature k, we have:

- The dynamics is reduced to dimension $n \leq 3$.
- The curvature must be non-negative, $k \geq 0$, so it does not make sense in, for example, the hyperbolic space.
- For $k = 0$ and up to topology, the dynamics actually occurs over arbitrary plane curves in $\mathbb{R}^3$. This, in particular, gives the answer to the above stated naive question.
- For $k > 0$, up to topology, the extremals over a round 3-sphere are just the so called Legendrian curves, that is, curves which are horizontal lifts, via the Hopf map, of curves in the 2-sphere.

Other partial results, providing examples and families of extremals in Berger spheres and in the complex projective plane, can be found in [1] (see also [4,6,7] for extremals in warped product spaces).

In this paper we will focus on this variational approach for curves in homogeneous 3-spaces whose isometry group has dimension four (sometimes called rotationally symmetric homogeneous 3-spaces [11]). In particular, we obtain the complete classification of extremals for the total curvature action in these backgrounds. The main tool to determine the extremals is the existence of an infinitesimal translation, ξ, which allows us to see each homogeneous 3-space with rigidity of order four as a fibration over a surface with constant curvature, which works, in turn, as the space of ξ-orbits. As a consequence, a congruence class