J. Math. Anal. Appl. 431 (2015) 509–517

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Invariant domains in the Hardy space over the unit disk

Li Chen [a], Guozheng Cheng [b,*]

[a] Department of Mathematics, Tianjin University, Tianjin, 300072, China
[b] School of Mathematics, Wenzhou University, Wenzhou 325035, China

A R T I C L E I N F O

Article history:
Received 21 January 2015
Available online 22 May 2015
Submitted by J.A. Ball

Keywords:
Invariant domain
Hardy space
Graph invariant subspace

A B S T R A C T

Inspired by some recent developments in operator theory on vector-valued function spaces, we identify and study a class of linear manifolds called invariant domains in Hilbert spaces, which implements a natural generalization of invariant subspaces for bounded operators. For coordinate multiplication operator on the Hardy space over the unit disc, a Beurling-type characterization of invariant domains together with some algebraic properties is obtained. We also go beyond the Hardy space with some related discussions on the structure of graph invariant subspaces (GIS).

1. Introduction

The study of closed invariant subspaces of the Hardy space $H^2(\mathbb{D})$ with respect to the multiplication operator M_z forms a beautiful chapter in operator theory. In particular, Beurling's theorem has been the source of inspiration for many important works. The purpose of this paper is to introduce and study a new class of M_z invariant linear subspaces, not necessarily closed but including those closed ones, called *invariant domains*.

As function theory plays a dominating role in the scalar-valued function spaces, more techniques are developed to study invariant subspaces in vector-valued spaces. A closed linear subspace $\mathcal{M}$ of $H \otimes \mathbb{C}^N$, where H is a Hilbert space of holomorphic functions with point-wise evaluation, is called an *invariant subspace* if it is invariant under $M_z \otimes I$ where I denotes the identity operator on $\mathbb{C}^N$. Our study of invariant domain makes a reasonable connection between the operator theory in H and in $H \otimes \mathbb{C}^N$, and is motivated, in a large part, by the *transitive algebra problem* which extends the well-known invariant subspace problem. We refer the readers to [7] for a nice introduction to the transitive algebra problem and some related topics.

Arveson [1] reduces the transitive algebra problem to the question of determining the structure of a special kind of invariant subspaces in $H \otimes \mathbb{C}^N$, called *graph invariant subspaces* (GIS) (see Section 5 for more details).

* Corresponding author.
E-mail addresses: lchencz@tju.edu.cn (L. Chen), chgzh09@gmail.com (G. Cheng).

http://dx.doi.org/10.1016/j.jmaa.2015.05.039
0022-247X/

Definition 1. An invariant subspace $\mathcal{M}$ of $H \otimes \mathbb{C}^N (N \geqslant 2)$ is called a **graph invariant subspace (GIS)** if there exist linear transformations $T_1, \cdots, T_{N-1}$, not necessarily bounded, defined on a nonzero linear manifold D in H, such that

$$\mathcal{M} = \{(f, T_1 f, \cdots, T_{N-1} f) : f \in D\}$$

with respect to a fixed base $\{e_1, \cdots, e_N\}$ of $\mathbb{C}^N$.

Since $\mathcal{M}$ is invariant under M_z, D is a z-invariant linear manifold and each T_i, called a *graph transformation*, commutes with M_z. It is easy to see, by the closed graph theorem, that D is closed if and only if every graph transformation is bounded. Apart from the transitive algebra problem which will be discussed in more details in later sections, GIS serves as an effective tool to study unbounded operators on function spaces and we refer to [9] for a discussion of closed operators via their graphs on the Hardy space.

Remark 2. One can define in a more generalized way so-called k-graph invariant subspaces, which are invariant subspaces in $H \otimes \mathbb{C}^N (N \geqslant 2)$ of the form

$$\mathcal{M} = \{(f_1, \cdots, f_k, T_1(f_1, \cdots, f_k), \cdots, T_{N-k}(f_1, \cdots, f_k)) : (f_1, \cdots, f_k) \in D\}$$

where D is a linear manifold in $H \otimes \mathbb{C}^k$ and $T_1, \cdots, T_{N-k}$ are linear transformations on the first k coordinates. Every invariant subspace in $H \otimes \mathbb{C}^N$ can be viewed as a k-graph invariant subspace where $1 \leq k \leq N$. In particular, it is an N-graph invariant subspace if no such graph transformations exist. In this paper we only consider GIS, or 1-GIS and it is sufficient for the transitive algebra problem on H, as we will illustrate in Section 5.

Definition 3. A linear manifold D in H is called an invariant domain if there exists a GIS $\mathcal{M} \subseteq H \otimes \mathbb{C}^N$ such that

$$\mathcal{M} = \{(f, T_1 f, \cdots, T_{N-1} f) : f \in D\}.$$

Coming from invariant subspaces in $H \otimes \mathbb{C}^N$, the invariant domain is a reasonable generalization of invariant subspaces in H. First observe that any invariant subspace in H is automatically an invariant domain, corresponding to a GIS with bounded graph transformations commuting with M_z. We also point out that invariant domains, though not necessarily closed, are *operator ranges*. Operator ranges by definition are ranges of bounded operators and enjoy certain properties of closed subspaces distinguishing them from arbitrary linear manifolds (see [3]).

To be precise, to a nonzero vector e in $\mathbb{C}^N$ and an invariant subspace $\mathcal{M} \subseteq H \otimes \mathbb{C}^N$, one can associate a natural operator range $P_{H \otimes e}\mathcal{M}$, where $P_{H \otimes e}$ denotes the projection onto $H \otimes e$, called the *slice* of $\mathcal{M}$ respect to e (or *e-slice* for short). Since $\mathcal{M}$ is invariant under $M_z \otimes I$, the slice $P_{H \otimes e}\mathcal{M}$, which can be identified with a linear manifold of H in an obvious way, is invariant under M_z. In particular, the invariant domain of a graph invariant subspace $\mathcal{M}$ is the slice $P_{H \otimes e_1}\mathcal{M}$. Besides invariant domains, we will also present some results on how slices are related to the structure of invariant subspaces in $H \otimes \mathbb{C}^N$.

The present paper begins with a Beurling-type characterization of invariant domains in the Hardy space $H^2(\mathbb{D})$. The characterization enables us to study operations on invariant domains in $H^2(\mathbb{D})$, which is partially inspired by corresponding results for operator ranges in [3]. We complement the study on invariant domains by passing to general slices on an arbitrary function space and relate it to the transitive algebra problem.