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Uniqueness of limit cycles for sewing planar piecewise linear systems

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A B S T R A C T

This paper proves the uniqueness of limit cycles for sewing planar piecewise linear systems with two zones separated by a straight line, Σ, and only one Σ-singularity of monodromic type. The proofs are based in an extension of Rolle's Theorem for dynamical systems on the plane.

1. Introduction and main results

Usually the models used in many problems related to engineering, biology, control theory, design of electric circuits, mechanical systems, economics science, and medicine are differential systems that are neither analytic, nor differentiable. A good tool to describe the dynamics of these models is the study of piecewise differential systems. See [1,8] for a wide selection of models and real applications. Typically this class of systems is obtained using two or more linear vector fields that are defined on different regions separated by discontinuity boundaries. In particular, a circuit having an ideal switch with state feedback can be modeled with a planar piecewise linear system where the discontinuity boundary is defined by a straight line, see Sec. 1.1.7 of [1].

Planar linear differential systems are completely understood using only linear algebra and they do not present isolated periodic orbits, so-called *limit cycles*. This is not the case for piecewise linear differential systems. The classification of the different phase portraits or the study of the maximum number of limit cycles are still open problems, even when the number of regions is small, two in our case, or the boundaries are straight lines. The existence of real and/or virtual singularities, connection of separatrices, isolated periodic orbits, ... increase, in comparison with the linear one, the number of possible phase portraits in the

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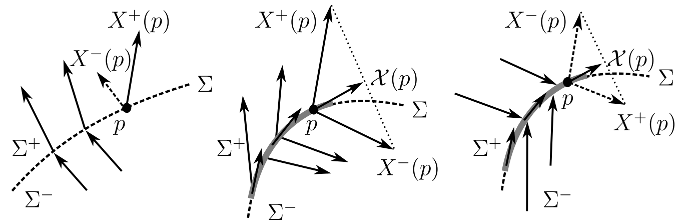

Fig. 1. Definition of the vector field on Σ following Filippov's convention in the sewing, escaping, and sliding regions.

class of piecewise linear differential systems. We focus our attention only on the number of isolated periodic solutions. More concretely, we focus on the maximum number of limit cycles that a class of piecewise linear systems can present. This is a very intricate problem and there are few studies that provide a complete answer, even when we restrict it for a concrete class of planar systems. It can be considered as a generalization of the classical 16th Hilbert problem, see [16]. In our approach the study of uniqueness of limit cycles relates to the uniqueness of intersection points of algebraic curves. One of the main tools used in this paper is the extension of Rolle's Theorem for dynamical systems on the plane, introduced by Khovanskii in 1982, see [17].

Lum and Chua conjectured in 1990 that continuous piecewise linear systems in two zones have at most one limit cycle, see [21]. This was proved in [11]. Few years ago, as an application of a Liénard criterion, another piecewise linear family with at most one limit cycle was studied, see [20]. All these cases have no sliding regions. Recently, examples of piecewise linear systems with a sliding segment that present two and three limit cycles are provided, see among others [3,13–15] and [5,7,19], respectively.

Before explaining our results with more detail we introduce some notation. Let 0 be a regular value of a function $h : \mathbb{R}^2 \to \mathbb{R}$. We denote the discontinuity boundary by $\Sigma = h^{-1}(0)$ and the two regions by $\Sigma^\pm = \{\pm h(x,y) > 0\}$. With this notation we consider the Σ-piecewise vector field $\mathcal{X} = (X^+, X^-)$ defined by

$$\mathcal{X}(q) = \begin{cases} X^+(q), & h(q) > 0, \\ X^-(q), & h(q) < 0, \end{cases}$$

where $X^\pm$ are planar vector fields. The singularities $p^\pm$ of $X^\pm$ are called *real* or *virtual* if $p^\pm \in \Sigma^\pm$ or $p^\pm \in \Sigma^\mp$, respectively.

The vector field $\mathcal{X}$ is defined on Σ following Filippov's terminology, see [10] and Fig. 1. The points in Σ where both vectors fields $X^\pm$ simultaneously point outward or inward from Σ define the *escaping* or *sliding region*, the complement in Σ defines the *sewing region*. In fact the boundary of both regions is defined by the tangential points of $X^\pm$ in Σ. The *sewing points* in $\Sigma = h^{-1}(0)$ of $\mathcal{X} = (X^+, X^-)$ satisfy $X^+h(p) \cdot X^-h(p) > 0$, where X^+h denote the derivative of the function h in the direction of the vector X^+ i.e., $X^+h(p) = \langle \nabla h(p), X^+(p) \rangle$. Equivalently for X^-h.

The point p in Σ is a *tangential point* of $X^\pm$ if $X^\pm h(p) = 0$ and we say that p is a Σ-*singularity* of $\mathcal{X}$ if $p \in \Sigma$ and it is either a tangential point or a singularity of X^+ or X^-. We call $p \in \Sigma$ an *invisible fold* of $X^\pm$ if p is a tangential point of $X^\pm$ and $(X^\pm)^2 h(p) < 0$. Moreover, $p \in \Sigma$ is a Σ-*monodromic singularity* of $\mathcal{X} = (X^+, X^-)$ if p is a Σ-singularity of $\mathcal{X}$ and there exists a neighborhood of the p such that the solutions of $\mathcal{X}$ turn around it either in forward or in backward time. See Fig. 2.

The objective of this paper is to prove the uniqueness of limit cycle for a class of planar piecewise linear systems where Σ is formed by sewing regions with a unique Σ-singularity that is of monodromic type. As the vector fields $X^\pm$ are linear, all the isolated periodic solutions should cross the discontinuity boundary Σ.

More concretely, we consider $\mathcal{X} = (X^+, X^-)$ and $\Sigma = h^{-1}(0)$, where $X^\pm$ are linear vector fields, h is also linear and p is the unique Σ-monodromic singularity. The vector field $\mathcal{X}$, after a rotation and a translation if necessary, can be expressed by