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Global existence for semi-linear structurally damped σ-evolution models

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A B S T R A C T

The main purpose of this paper is to study the global existence of small data solutions for semi-linear structurally damped σ-evolution models of the form

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = f(|D|^a u, u_t),\ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x)$$

with $\sigma \geq 1$, $\mu > 0$ and $\delta = \frac{\sigma}{2}$. This is a family of structurally damped σ-evolution models interpolating between models with exterior damping $\delta = 0$ and those with visco-elastic type damping $\delta = \sigma$. The function $f(|D|^a u, u_t)$ represents power nonlinearities $||D|^a u|^p$ for $a \in [0, \sigma)$ or $|u_t|^p$. Our goal is to propose a Fujita type exponent diving the admissible range of powers p into those allowing global existence of small data solutions (stability of zero solution) and those producing a blow-up behavior even for small data. On the one hand we use new results from harmonic analysis for fractional Gagliardo–Nirenberg inequality or for superposition operators (see Appendix A), on the other hand our approach bases on $L^p - L^q$ estimates not necessarily on the conjugate line for solutions to the corresponding linear models assuming additional L^1 regularity for the data. The linear models we have here in mind are

$$v_{tt} + (-\Delta)^\sigma v + \mu(-\Delta)^\delta v_t = 0,\ v(0,x) = v_0(x),\ \ v_t(0,x) = v_1(x)$$

with $\sigma \geq 1$, $\mu > 0$ and $\delta \in (0, \sigma]$.

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1. Introduction

Semi-linear wave models with power non-linearity of the form

$$u_{tt} - \Delta u = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x)$$

are discussed in a lot of papers. It was a long and interesting way to prove Strauss' conjecture. Later colleagues understood the benefit and influence of a damping term in the wave model for the global existence of small data solutions (see, for example, [13,21,9] and the references therein). The model of interest is

$$u_{tt} - \Delta u + bu_t = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x).$$

A next challenge was to include a time dependent coefficient $b = b(t)$ in the dissipation term. Here [16] and [12] considered the question for a Fujita exponent (global existence of small data solutions via blow-up) for the case $b(t) = (1+t)^\beta$, $\beta \in (-1,1)$. From the proposed classifications of dissipations in [23] and [24] it follows that [16] considered a special family of effective damping terms. Thus, it was reasonable to generalize the results from [16] to the family of effective damping terms $b(t)u_t$. This is done in [6]. The next step was to use other damping mechanisms, for example, structural damping terms of the form $(-\Delta)^\delta u_t$. In the paper [7] the authors discussed Fujita type exponents for semi-linear structurally damped wave models with power non-linearity of the form

$$u_{tt} - \Delta u + \mu(-\Delta)^\delta u_t = |u|^p, \; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \tag{1}$$

with $\mu > 0$ and $\delta \in (0,1]$. New strategies are proposed to handle these models. Depending on the values of δ the family of models was divided into parabolic like models $\delta \in (0, \frac{1}{2})$ and hyperbolic like models $\delta \in (\frac{1}{2}, 1]$. The "critical value" is $\delta = \frac{1}{2}$. Then mixing of regularity of data, higher regularity of data and $L^p - L^q$ estimates of solutions to the corresponding linear models with p and q not necessarily from the conjugate line are new strategies which are used in different approaches. The goal of the present paper is to propose a new research topic. We want to study semi-linear σ-evolution models ($\sigma = 1$ implies semi-linear wave models) in the critical case $\delta = \frac{\sigma}{2}$, but now with new power non-linearities $||D|^a u|^p$, $a \in [0,\sigma)$, or $|u_t|^p$. We see that the first power non-linearity contains a pseudo-differential action. The second power non-linearity is related to a term from the usual energy. The models we have in mind are

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = ||D|^a u|^p, \; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{2}$$

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = |u_t|^p, \; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{3}$$

The operators $(-\Delta)^a$ or $|D|^a$, $a > 0$, are defined as follows:

$$(-\Delta)^\alpha u = F^{-1}(|\xi|^{2a} F(u)), \;\; |D|^a u = F^{-1}(|\xi|^a F(u)) \;\text{ for all }\; u \in H^{-\infty}(\mathbb{R}^n).$$

Our approach combines tools from linear and non-linear theory. On the one hand we should understand how to handle the non-linear terms (even with pseudo-differential actions) in scales of Besov spaces. Here we apply (recently proved) results from harmonic analysis (see Appendix A). On the other hand we know from the paper [7], that $L^p - L^q$ estimates not necessarily on the conjugate line for the corresponding linear models

$$v_{tt} + (-\Delta)^\sigma v + \mu(-\Delta)^\delta v_t = 0, \; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x) \tag{4}$$