J. Math. Anal. Appl. 431 (2015) 597–621

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Compactness for the commutators of singular integral operators with rough variable kernels

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ARTICLE INFO

Article history:
Received 25 December 2014
Available online 3 June 2015
Submitted by R.H. Torres

Keywords:
Commutator
Singular integral operator with variable kernel
Maximal operator
Compact operator

ABSTRACT

Let T_Ω be the singular integral operator with variable kernel defined by

$$T_\Omega f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{\Omega(x,\, x-y)}{|x-y|^n} f(y)\mathrm{d}y,$$

where $\Omega(x, y)$ is homogeneous of degree zero in the second variable y, and $\int_{S^{n-1}} \Omega(x, z')\mathrm{d}\sigma(z') = 0$ for any $x \in \mathbb{R}^n$. In this paper, the authors prove that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$, then the commutator generated by a $\mathrm{CMO}(\mathbb{R}^n)$ function and T_Ω, and the associated lacunary maximal operator, are compact on $L^2(\mathbb{R}^n)$. The associated continuous maximal commutator is also considered.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 2$. Let $\Omega(x, z)$ be a function on $\mathbb{R}^n \times \mathbb{R}^n$, which is homogeneous of degree zero with respect to the variable z. Throughout this paper, for such a function $\Omega(x, z)$, we assume that Ω satisfies the vanishing condition that

$$\int_{S^{n-1}} \Omega(x, z')\mathrm{d}\sigma(z') = 0 \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jmaa.2015.05.081
0022-247X/

for any $x \in \mathbb{R}^n$, here and in what follows, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, and for $z \in \mathbb{R}^n$, $z' = z/|z|$. For a fixed $q \in [1, \infty]$, we say that $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, if

$$\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} = \sup_{x\in\mathbb{R}^n} \Big(\int_{S^{n-1}} |\Omega(x, z')|^q \mathrm{d}\sigma(z') \Big)^{1/q} < \infty.$$

The singular integral operator with variable kernel, associated with Ω, is defined by

$$T_\Omega f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\mathrm{d}y$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$. These operators were introduced by Calderón and Zygmund in their celebrated works [5,6], and are relevant in second order linear elliptic equations with variable coefficients. Calderón and Zygmund [5,6] proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$, then T_Ω is bounded on $L^2(\mathbb{R}^n)$ and the condition $q > 2(n-1)/n$ is optimal in the sense that the $L^2(\mathbb{R}^n)$ boundedness of T may fail if $q \le 2(n-1)/n$. Moreover, Calderón and Zygmund [7] showed that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $p \in (2, \infty)$ if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ with $\frac{1}{q} < \frac{1}{p}\frac{n}{n-1} + \big(1 - \frac{2}{p}\big)$.

The maximal singular integral operator associated with T_Ω is given by

$$T^\star_\Omega f(x) = \sup_{\epsilon>0} \Big| \int_{|x-y|\ge\epsilon} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\mathrm{d}y \Big|.$$

This operator plays an important role in studying the almost everywhere convergence of the singular integral. Aguilera and Harboure [1] proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 4(n-1)/(2n-1)$, then $T^\star_\Omega$ is bounded on $L^2(\mathbb{R}^n)$. Cowling and Mauceri [15], Christ, Duoandikoetxea and Rubio de Francia [13], proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$, then $T^\star_\Omega$ is bounded on $L^2(\mathbb{R}^n)$.

Now let $b \in \mathrm{BMO}(\mathbb{R}^n)$, the space of functions of bounded mean oscillation which was introduced by John and Nirenberg. The commutator generated by b and T_Ω is defined by

$$T_{\Omega,\,b} f(x) = b(x) T_\Omega f(x) - T_\Omega(bf)(x), \tag{1.2}$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$. We define the maximal operator $T^\star_{\Omega,\,b}$, corresponding to $T_{\Omega,\,b}$, by

$$T^\star_{\Omega,\,b} f(x) = \sup_{\epsilon>0} \Big| \int_{|x-y|>\epsilon} \big(b(x) - b(y)\big) \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\mathrm{d}y \Big|. \tag{1.3}$$

Chiarenza et al. [12] proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times C^\infty(S^{n-1})$, then $T_{\Omega,\,b}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Di Fazio and Ragusa [17] considered the boundedness of $T_{\Omega,\,b}$ on Morrey spaces. By subtle Fourier transform estimates and Littlewood–Paley theory, Chen and Ding [8] proved that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$ is sufficient for $T_{\Omega,\,b}$ to be bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Furthermore, Chen, Ding and Li [9] showed that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$ also implies that $T^\star_{\Omega,\,b}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

Uchiyama [20] considered the compactness of the commutator of singular integral operators. Let $\mathrm{CMO}(\mathbb{R}^n)$ be the closure of $C_0^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincides with $\mathrm{VMO}(\mathbb{R}^n)$, the space of functions of vanishing mean oscillation introduced by Coifman and Weiss in [14], see also [4]. Uchiyama proved that if S is a Calderón–Zygmund operator, and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, S]$, the commutator of S and b, as in (1.2), is a compact operator on $L^p(\mathbb{R}^n)$ $(p \in (1, \infty))$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$. This shows that for $\mathrm{CMO}(\mathbb{R}^n)$ functions b, the properties of $[b, S]$ maybe better than that of the operator S. Since then,