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Whitney's Theorem: A nonsmooth version ☆

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ABSTRACT

In this paper we solve the problem of extending continuous functions with nonempty subdifferential at every point of a closed subset A of $\mathbb{R}^n$ to functions with the same property defined in the whole $\mathbb{R}^n$, keeping the property of outer semicontinuity of the subdifferential, which is a set-valued function. The proof is constructive, and gives us a wide range of possible extensions.

1. Introduction and preliminary results

Extension problems play a central role in analysis. They take the following general form. Let X and Y be metric spaces, let $F(X,Y)$ be a class of functions from X to Y, and $A \subset X$. The extension problem is to determine whether each function in $F(A,Y)$ has an extension in $F(X,Y)$. The most celebrated classical results are the theorem of Tietze, for continuous functions, and the Whitney's Extension Theorem, for C^m functions [7]. But we cannot forget the simple formula for extending real-valued Lipschitz functions by means of inf-convolution that McShane, and Whitney himself, pointed out. When we are looking for extensions, it is important that the extension keeps as many properties of the original function as possible. Moreover, it is also important to know if we are able to construct such extensions, or we can only guarantee their existence.

Nonsmooth analysis is becoming a more and more important tool to deal with functions, because it allows us to extend Calculus to a broader setting. Among the different nonsmooth concepts the subdifferential is probably the most important. For an elementary discussion about this subject see [5].

The natural frame for subdifferentiability is that of real extended valued lower semicontinuous functions. However, if we want to extend functions, non-continuous extensions seem of little interest. In fact any bounded continuous function can be extended to a lower semicontinuous function in a trivial way, moreover if we admit extended valued functions, any continuous function f on a closed subset A of $\mathbb{R}^n$ can be extended

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to $\mathbb{R}^n$ as a lower semicontinuous function, by setting $f = +\infty$ on $\mathbb{R}^n \setminus A$. For this reason we are interested only in continuous functions and continuous extensions.

Let us recall the definition of the subdifferential.

Definition 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$. We define the subdifferential of f at x_0 as the set of $\zeta \in \mathbb{R}^n$ satisfying:

$$\liminf_{x \to x_0} \frac{f(x) - f(x_0) - \langle \zeta, x - x_0 \rangle}{\|x - x_0\|} \geq 0.$$

We will denote this subdifferential by $\partial f(x_0)$. We will use the following characterization:

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$. Then $\zeta \in \partial f(x_0)$ if and only if there exists a differentiable function φ such that $\varphi(x_0) = f(x_0)$, $\nabla \varphi(x_0) = \zeta$, and $\varphi(x) \leq f(x)$ near x_0.*

Another useful notion is the limiting subdifferential, denoted by $\partial_L f(x_0)$.

Definition 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $x_0 \in \mathbb{R}^n$, and $\zeta_0 \in \mathbb{R}^n$. We say that $\zeta_0 \in \partial_L f(x_0)$ provided that there are sequences $(x_n)_n$, converging to x_0, and $(\zeta_n)_n$, converging to ζ_0, such that $\zeta_n \in \partial f(x_n)$.

If a function f satisfies $\partial_L f(x_0) = \partial f(x_0)$, we say that f is regular at x_0. We will also use the following notion.

Definition 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, $x_0 \in \mathbb{R}^n$. We define the generalized gradient of f at x_0, denoted by $\overline{\nabla} f(x_0)$, as the closed convex hull of the set $\partial_L f(x_0)$.

From the fact that $\partial f(x_0)$ is always a closed convex set, it follows that for locally Lipschitz regular functions we have

$$\overline{\nabla} f(x_0) = \partial_L f(x_0) = \partial f(x_0).$$

We now present some elementary results whose proofs we will omit.

Proposition 2. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two functions. Suppose that f is C^1 and non-negative, and g is lsc. Then*

$$\partial(fg)(x_0) = g(x_0)\nabla f(x_0) + f(x_0)\partial g(x_0)$$

for every x_0 such that $f(x_0) > 0$.

Proposition 3. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two functions. Suppose that f is C^1 and non-negative, and g is continuous. Then*

$$\partial_L(fg)(x_0) = g(x_0)\nabla f(x_0) + f(x_0)\partial_L g(x_0)$$

for every x_0 such that $f(x_0) > 0$.

Corollary 4. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two functions. Suppose that f is C^1 and non-negative, and g is continuous and regular. Then fg is also regular at every x_0 such that $f(x_0) > 0$.*