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# Spectral properties of truncated Toeplitz operators by equivalence after extension

M. Cristina Câmara [a], Jonathan R. Partington [b,*]

[a] Center for Mathematical Analysis, Geometry, and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

[b] School of Mathematics, University of Leeds, Leeds LS2 9JT, UK

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A B S T R A C T

We study truncated Toeplitz operators in model spaces $K_\theta^p$ for $1 < p < \infty$, with essentially bounded symbols in a class including the algebra $C(\mathbb{R}_\infty) + H_\infty^+$, as well as sums of analytic and anti-analytic functions satisfying a $\theta$-separation condition, using their equivalence after extension to Toeplitz operators with $2 \times 2$ matrix symbols. We establish Fredholmness and invertibility criteria for truncated Toeplitz operators with $\theta$-separated symbols and, in particular, we identify a class of operators for which semi-Fredholmness is equivalent to invertibility. For symbols in $C(\mathbb{R}_\infty) + H_\infty^+$, we extend to all $p \in (1, \infty)$ the spectral mapping theorem for the essential spectrum. Stronger results are obtained in the case of operators with rational symbols, or if the underlying model space is finite-dimensional.


## 1. Introduction

This paper is concerned with truncated Toeplitz operators (TTO), a natural generalisation of finite Toeplitz matrices; these have received much attention since they were introduced by Sarason [25]: see, for instance, [1] and the recent survey [17]. They are encountered in various contexts, for example in the study of finite Toeplitz matrices and finite-time convolution operators.

By using the equivalence after extension of TTO to block Toeplitz operators of a particular form [10], the corona theorem, and the solutions to certain associated Riemann–Hilbert problems, we study here the invertibility and Fredholmness of several classes of TTO, together with their spectra and essential spectra.

Here our context is the Hardy space $H_p^+$ of the upper half-plane for $1 < p < \infty$, rather than simply $H_2^+$. Considering different values of $p$ in $(1, \infty)$ naturally requires new approaches to the study of TTO, providing alternatives to Hilbert space methods. By doing so, we not only obtain various results that are new even

* Corresponding author.
*E-mail addresses:* ccamara@math.ist.utl.pt (M.C. Câmara), j.r.partington@leeds.ac.uk (J.R. Partington).

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for $p = 2$, but we also shed light on whether the properties that are studied, namely spectral properties of TTO, depend on the existence of an underlying Hilbert space structure, or on the value of $p$.

In fact, properties such as Fredholmness, invertibility and the dimensions of the kernels and the cokernels of Toeplitz operators in the Hardy spaces $H_p^+$ may depend on the value of $p \in (1, \infty)$; it is easy to find examples of this behaviour by considering piecewise continuous symbols of the form $g_\alpha(\xi) = (\frac{\xi - i}{\xi + i})^\alpha$ [12,20,22]. One would expect the same to hold for TTO defined in a model space $K_\theta^p := H_p^+ \cap \theta H_p^-$, where $\theta$ is an inner function; however, somewhat surprisingly, the results obtained for the various classes of TTO considered in this paper do not depend on $p$. Note however, that in general the space $K_\theta^p$ on which the TTO are defined *does* depend on $p$: see, for example [8,14]. For example, this is the case for any infinite Blaschke product $\theta$ whose zeroes are not bounded away from the real axis. Thus the kernel of a TTO will in general depend on $p$.

We first consider here TTO with essentially bounded symbols of the form

$$g = \bar{\theta}_1 \, a_- + \theta_2 \, a_+ \quad , \quad a_\pm \in \mathcal{M}_\infty^\pm ,$$

where, denoting by $\mathcal{R}$ the set of all rational functions in $L_\infty(\mathbb{R})$, $\mathcal{M}_\infty^\pm := H_\infty^\pm + \mathcal{R}$ and $\theta_1$ and $\theta_2$ are inner functions such that $\theta$ divides $\theta_1 \theta_2$. An important property of this class of TTO is that it is possible to determine a solution to an associated Riemann–Hilbert problem, which makes it easier to study; in fact, the study of general TTO presents great difficulties. Moreover this class of symbols, which we call $\theta$-separated, includes all functions in $H_\infty^+ \cup H_\infty^- \cup \mathcal{R}$, and its study reveals some remarkable properties and raises new questions.

For bounded analytic symbols we determine the spectrum of TTO on $K_\theta^p$ for each $p \in (1, \infty)$, a result previously established only for $p = 2$ (Fuhrmann's extension [16] of the Livšic–Moeller theorem [21,23,24]). The results obtained for symbols in $\mathcal{M}_\infty^+$ allow us to describe the essential spectra of TTO with symbols in $C(\mathbb{R}_\infty) + H_\infty^+$, extending Bessonov's results [5] to TTO acting on $K_\theta^p$ for all $p \in (1, \infty)$.

Furthermore, for rational symbols we establish necessary and sufficient conditions for invertibility of the associated TTO, which enables us to give a more geometric description of the point spectrum and the spectrum of a TTO whose symbol $R$ admits only one pole, and to obtain an explicit expression for the resolvent operator $(A_R^\theta - \lambda I)^{-1}$ if $\lambda \notin \sigma(A_R^\theta)$.

Finally, for TTO defined in finite-dimensional model spaces (in which case the space does not depend on $p$), we characterise the operator's kernel and invertibility properties, and we illustrate the results by giving a simple description of the eigenvalues and the corresponding eigenspaces of a TTO defined in a model space with dimension 2. Those results show in particular that, while the general case of TTO with discontinuous symbols of the form $g_\alpha$ mentioned above is yet to be fully investigated, in the particular case where the model space is defined by a finite Blaschke product the dimensions of the kernel and the cokernel of a TTO with a symbol of that type (or any other symbol in $L_\infty$) do not depend on $p$. This is not the case for more general model spaces, as we show in Example 3.6.

The paper is organised as follows. The equivalence after extension of TTO to block Toeplitz operators of a particular form is explained in Section 2, along with the remaining preliminary material. In Section 3 we discuss a class of TTO with $\theta$-separated symbols, and analyse their kernels and their Fredholm properties. Section 4 is concerned with analytic symbols, and Section 5 with $C(\mathbb{R}_\infty) + H_\infty^+$ (and, in particular, rational) symbols. Finally, in Section 6 we consider the case when the underlying model space is finite-dimensional.

## 2. Preliminaries

For $1 \leq p \leq \infty$ we let $H_p^\pm$ denote the Hardy spaces of the upper and lower half-planes, recalling that for $1 < p < \infty$ we have the decomposition $L_p(\mathbb{R}) = H_p^+ \oplus H_p^-$ with associated projections $P_+$ and $P_-$. In what