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Linear stability of double–double orbits in the parallelogram four-body problem

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ABSTRACT

We study the linear stability of a two-parameter family of periodic orbits in the parallelogram four-body problem. This family was numerically found and named as double–double orbits by Vanderbei. A demonstration of such an orbit is shown. By introducing new transformations and applying Roberts' symmetry reduction method, the linear stability can be simplified to the calculation of two eigenvalues. A global picture of linear stability of this set with respect to the two parameters is given for the first time. Actually, most of the double–double orbits are numerically proved to be linearly stable.

1. Introduction and main result

In 2004, Vanderbei [8] successfully applied his minimizing program to the N-body problem and found many new periodic orbits. In his list, there is an interesting set of orbits in the parallelogram four-body problem with equal-masses, which is apparently stable and named as double–double orbit. In such an orbit, two pairs of bodies locate on two sides of the origin. In each pair, the two bodies switch orders periodically. For example, as in Fig. 1(a) or (b), at $t = 0$, the four bodies line up on the x-axis with an order $m_1 \to m_2 \to m_3 \to m_4$ from right to left. At $t = 1$, they are on a line which is an anti-clockwise rotation of θ of the x-axis. The order of the bodies becomes $m_2 \to m_1 \to m_4 \to m_3$ from right to left. At $t = 2$, they line up for the third time and the order becomes $m_1 \to m_2 \to m_3 \to m_4$ from right to left, which is the same as it is at $t = 0$. Actually, the four bodies always form a parallelogram for $t \in [0, 2]$. The existence of this set of double–double orbits is given by Chen [3]. The linear stability of some of the double–double orbits is claimed by Vanderbei [8]. Actually, by testing several double–double orbits on a simulator for a

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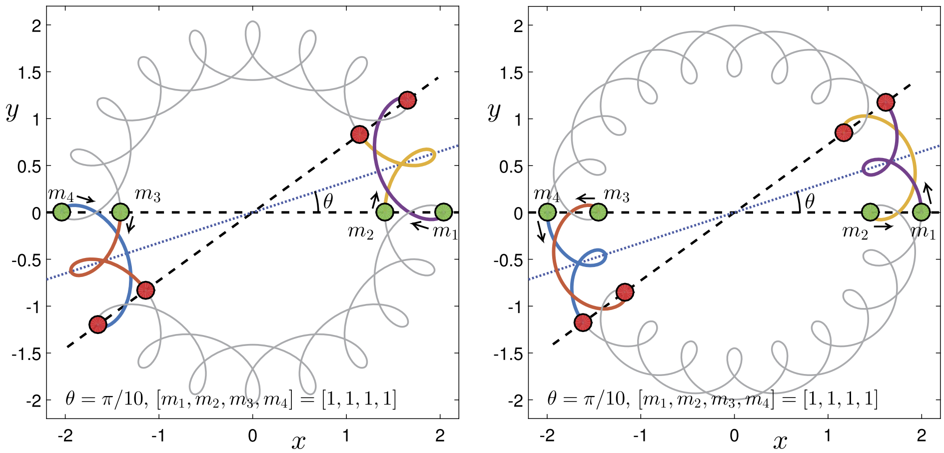

(a) Retrograde double-double orbit (b) Prograde double-double orbit

Fig. 1. Two types of double–double orbits with equal masses: the one on the left is called **retrograde double–double orbit**; the one on the right is **prograde double–double orbit**. In each graph, θ is the angle between the first collinear configuration at $t = 0$ and the second collinear configuration at $t = 1$. Both pictures are for $\theta = \pi/10$. The highlighted parts are the orbits for $t \in [0, 2]$.

relatively long time, he concluded that they are apparently stable. However, the linear stability of this set is not clear.

In this paper, we extend the double–double orbits to the general mass case $M = [1, m, m, 1]$ with $m \in (0, 1]$ and analyze their linear stabilities with respect to the two parameters: mass m and rotation angle θ (as in Fig. 1). A new variational approach is introduced to search for the double–double orbits. At $t = 0$, we let the four bodies start with a collinear configuration on the x-axis. At $t = 1$, the four bodies line up for the second time. This line is an anti-clockwise rotation of θ of the x-axis (as in Fig. 1). The orbit basically repeats this movement and it becomes periodic if θ/π is rational. Then we have a two-parameter family of orbits: mass m and rotation angle θ. Geometrically, if $\theta = \pi/n$ with n an integer, the orbit has period $2n$. The graph of a double–double orbit has $4n$ loops if n is odd and $2n$ loops if n is even. For example, as in Fig. 1, $\theta = \pi/10$, and the orbit has 20 loops. (Actually, in Fig. 1, m_1 and m_4 share a trajectory with 10 loops. Similarly, the trajectory of m_2 and m_3 has another 10 loops.) Let $m \in (0, 1]$ with a step length 0.1. In order to see a clear picture of this two-parameter family, we only focus on special rotation angles $\theta = \pi/n$. For other rotation angles, the double–double orbits have similar motions. In fact, there are mainly two types of double–double orbits in this family: one is called **the retrograde double–double orbit**, in which the relative angular momentum of each double-pair and the total angular momentum of the four bodies are retrograde, as shown in Fig. 1(a); the other is called **the prograde double–double orbit**, in which the two angular momentums are prograde, as shown in Fig. 1(b). By introducing new transformations, we can eliminate all the first integrals except the Hamiltonian constant. The symmetry of double–double orbits is studied carefully. Roberts' symmetry reduction method [7] is then applied to eliminate the Hamiltonian constant and analyze the linear stability. A global picture of the linear stability of double–double orbits is given for the first time.

Let $\theta = \pi/n$ with integer $n \in [2, 20]$. Let $m \in (0, 1]$ with a step length 0.1. Both retrograde double–double orbits and prograde double–double orbits are linearly stable for any integer $n \in [4, 20]$ and mass $m \in (0, 1]$. However, for $n = 2$ and $n = 3$, the two types of double–double orbits have different linear stabilities. Based on the numerical calculation and the error analysis, the following theorem holds:

Theorem 1 *(Main theorem). Let $\theta = \pi/n$ with integer $n \in [2, 20]$. Let $m \in (0, 1]$ with a step length 0.1. The retrograde double–double orbits are linearly stable for all integers $n \in [2, 20]$ and mass $m \in (0, 1]$.*

The prograde double–double orbits are linearly stable for all integers $n \in [4, 20]$ and mass $m \in (0, 1]$. When $n = 3$, the prograde double–double orbit is linearly stable if $m \in [0.2, 1]$. If $n = 3$ and $m = 0.1$, the