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Stability of essential spectra of self-adjoint subspaces under compact perturbations

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ABSTRACT

This paper studies stability of essential spectra of self-adjoint subspaces (i.e., self-adjoint linear relations) under finite rank and compact perturbations in Hilbert spaces. Relationships between compact perturbation of closed subspaces and relatively compact perturbation of their operator parts are first established. This gives a characterization of compact perturbation in terms of difference between the operator parts of perturbed and unperturbed subspaces. It is shown that a self-adjoint subspace is still self-adjoint under either relatively bounded perturbation with relative bound less than one or relatively compact perturbation or compact perturbation with a certain additional condition. By using these results, invariance of essential spectra of self-adjoint subspaces is proved under relatively compact and compact perturbations, separately. As a special case, finite rank perturbation is discussed. The results obtained in this paper generalize the corresponding results for self-adjoint operators to self-adjoint subspaces.

1. Introduction

Perturbation problems are one of the main topics in both pure and applied mathematics. The perturbation theory of operators (i.e., single-valued operators) has been extensively studied and many elegant results have been obtained (cf., [8,10,18]). In particular, stability of spectra of self-adjoint operators under perturbation has received lots of attention. We shall recall several most well-known results about it. If a perturbation term is a symmetric (i.e., densely-defined and Hermitian) and relatively compact operator to a self-adjoint operator, then the essential spectrum of the self-adjoint operator is invariant (see [12, Theorem 8.15] or [18, Theorem 9.9]). However, it was shown that its absolutely continuous spectrum may disappear under this perturbation even though the perturbation term is very small by H. Weyl and later generalized by von Neumann [8, Chapter 10, Theorem 2.1]. But if the perturbation term is finite rank or more generally belongs to the trace class, then its absolutely continuous spectrum is invariant [8, Chapter 10, Theorems 4.3 and 4.4]. These results have been extensively applied to study of stability of spectra of symmetric linear differen-

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tial operators and bounded Jacobi operators (i.e., second-order bounded and symmetric linear difference operators) including Schrödinger operators that have a strong physical background.

Recently, it was found that minimal and maximal operators generated by symmetric linear difference expressions are multi-valued or non-densely defined in general even though the corresponding definiteness condition is satisfied (cf., [11,14]), and similar are those generated by symmetric linear differential expressions that do not satisfy the definiteness condition [9]. So the classical perturbation theory of operators is not available in this case. Partially due to the above reason, the study of non-densely defined or multi-valued linear operators has attracted a great deal of interests in near half a century.

In 1998, Cross introduced a concept of essential spectrum of a multi-valued linear operator (also called linear relations or subspaces) in a complex normed space in terms of the nullity and deficiency of its complete closure, and showed its stability under relative compact perturbation with certain additional conditions (see [4, Theorem VII.3.2]). In 2014, Wicox gave five distinct essential spectra of linear relations in Banach spaces in terms of semi-Fredholm properties, and showed their stability under relative compact perturbation with some additional conditions and under compact perturbation, separately, in [19] (see Remarks 5.1 and 5.2 for more detailed discussions about their relations with those obtained in the present paper).

In 2009, Azizov with his coauthors introduced concepts of compact and finite rank perturbations of closed subspaces in $X \times Y$ in terms of difference between orthogonal projections of $X \times Y$ to the subspaces, where X and Y are Hilbert spaces [2]. They proved that a closed subspace is a finite rank or compact perturbation of another closed subspace if and only if the difference between their resolvents is a finite rank or compact operator in the intersection of their resolvent sets in the case that this intersection is not empty and $X = Y$ [2, Corollaries 3.4 and 4.5]. Further, they studied stability properties of spectral points of positive and negative type and type π in the non-self-adjoint case under several kinds of perturbations in the Krein spaces [3].

Minimal operators or subspaces, generated by symmetric linear differential and difference expressions, are closed Hermitian operators or subspaces, and their self-adjoint extensions are self-adjoint operators or subspaces. Their resolvents can be expressed by corresponding Green functions. In the case that the differential and difference expressions are singular, their resolvents are complicated in general, and much more complicated when their orders (or dimensions) are higher because several boundary conditions or coupled boundary conditions are involved. Moreover, the Green functions are often expressed by solutions of the systems rather than by coefficients of the systems. Therefore, in some cases it is more convenient to give out a characterization of perturbation in terms of the operators or subspaces themselves rather than their resolvents. For the operator case, concepts of relatively compact, and finite rank and trace class perturbations were given in terms of the difference between perturbed and unperturbed operators (see Definition 2.3 and Lemma 2.4 for relatively compact perturbation of operators in Section 2, and we refer to [8,12,15,18] for more detailed discussions). We shall try to give a similar characterization of compact and finite rank perturbations for closed subspaces to that for operators in the present paper.

In this paper, we focus on the stability of essential spectra of self-adjoint subspaces (i.e., self-adjoint linear relations) under perturbations in Hilbert spaces. Here, the essential spectrum of a subspace is defined as the subset of its spectrum consisting of either accumulation points or isolated eigenvalues of infinite multiplicity (see Definition 2.1, and see Remark 2.1 for detailed discussions about relations of this definition with those given in [4,19]). And the definitions of compact and finite rank perturbations of closed subspaces given in [2] are used in the present paper.

Note that the spectrum and various spectra of a self-adjoint subspace, including point, discrete, essential, continuous, singular continuous, absolutely continuous and singular spectra, can be only determined by its operator part [13, Theorems 2.1, 2.2, 3.4 and 4.1]. So it is natural to take into account perturbations of operator parts of the unperturbed and perturbed subspaces. However, we find that a summation of closed subspaces, $T = S + A$, cannot imply a similar relation of their operator parts, $T_s = S_s + A_s$, in general (see Section 3 for a detailed discussion). In addition, in dealing with the summation of subspaces, one