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On different definitions of numerical range

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ABSTRACT

We study the relation between the intrinsic and the spatial numerical ranges with the recently introduced "approximated" spatial numerical range. As the main result, we show that the intrinsic numerical range always coincides with the convex hull of the approximated spatial numerical range. Besides, we show sufficient conditions and necessary conditions to assure that the approximated spatial numerical range coincides with the closure of the spatial numerical range.

1. Introduction

The concept of numerical range of an operator goes back to O. Toeplitz, who defined in 1918 the field of values of a matrix, a concept easily extensible to bounded linear operators on a Hilbert space. In the 1950s, a concept of numerical range of elements of unital Banach algebras was used to relate the geometrical and algebraic properties of the unit, starting with a paper by H. Bohnenblust and S. Karlin where it is shown that the unit is a vertex of the unit ball of the algebra, and was also used in the developing of Vidav's characterization of C^*-algebras. Later on, in the 1960s, G. Lumer and F. Bauer gave independent but related extensions of Toeplitz's numerical range to bounded linear operators on Banach spaces which do not use the algebraic structure of the space of all bounded linear operators. We refer the reader to the monographs by F. Bonsall and J. Duncan [3,4] and to sections §2.1 and §2.9 of the very recent book [5] by M. Cabrera and A. Rodríguez-Palacios for more information and background. Let us present the necessary definitions and notation. We will work with both real and complex Banach spaces. We write $\mathbb{K}$ to denote the base field ($= \mathbb{R}$ or $\mathbb{C}$) and $\mathrm{Re}(\cdot)$ to denote the real part in the complex case and just the identity in the real case. Given a Banach space X, S_X is its unit sphere, X^* is the topological dual space of X and $L(X)$ is the Banach algebra of all bounded linear operators on X. The *intrinsic numerical range* (or *algebra*

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numerical range) of $T \in L(X)$ is

$$V(T) := \{\Phi(T) \,:\, \Phi \in L(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\},$$

where Id denotes the identity operator on X. The *spatial numerical range* of T is given by

$$W(T) := \{x^*(Tx) \,:\, x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

These two ranges coincide in the case when X is a Hilbert space. For arbitrary Banach spaces, the equality

$$\overline{\mathrm{co}\, W(T)} = V(T)$$

is valid for all $T \in L(X)$ ($\mathrm{co}\, A$ denotes the convex hull of a set A). This equality allows to study algebra numerical ranges of operators without taking into account elements of the (wild) topological dual of the space of operators and, conversely, to get easier proofs of results on spatial numerical ranges.

The above equality has been extended to more general setting, as bounded uniformly continuous functions from the unit sphere of a Banach space to the space [9,13], but it is known that it is not possible to be extended to all bounded functions [12]. On the other hand, it is possible to define numerical ranges of operators (or functions) with respect to a fixed operator (or function) which plays the role of the identity operator, as it is done in [9]. For the intrinsic numerical range, the definition is immediate (see Definition 1.1), but the case of the spatial numerical range is more delicate (see Definition 1.2), as we may produce empty numerical ranges. Very recently, a definition of an "approximated" spatial numerical range has been introduced [1] for operators between different Banach spaces, which can be easily extended to bounded functions (see Definition 1.3) and which is never empty. Let us present the definitions of numerical ranges that we will use in the paper in full generality, that is, for bounded functions from a non-empty set into a Banach space. Given a Banach space Y and a non-empty set Γ, we write $\ell_\infty(\Gamma, Y)$ to denote the Banach space of all bounded functions from Γ into Y endowed with the sumpremum norm.

The first definition is the so-called intrinsic numerical range (with respect to a fixed function) which appeared, in different settings, with many names and many notations, since the 1960s (holomorphic functions [8], bounded uniformly continuous functions [9], bounded linear operators [3], bounded functions [12,13], among others). Also, it is nothing but a particular case of the so-called numerical range spaces (see [11] or [5, §2.1 and §2.9]).

Definition 1.1 *(Intrinsic numerical range).* Let Y be a Banach space and let Γ be a non-empty set. We fix $g \in \ell_\infty(\Gamma, Y)$ with $\|g\| = 1$. For every $f \in \ell_\infty(\Gamma, Y)$, the *intrinsic numerical range* of f relative to g is

$$V_g(f) := \{\Phi(f) \,:\, \Phi \in \ell_\infty(\Gamma, Y)^*,\ \|\Phi\| = \Phi(g) = 1\}.$$

Observe that if $\mathcal{M}$ is a closed subspace of $\ell_\infty(\Gamma, Y)$ containing g and f, then $V_g(f)$ can be calculated using only elements in the dual of $\mathcal{M}$ (by Hahn–Banach theorem), so it only depends on the geometry around f and g. This is why this numerical range is called "intrinsic". Let us also observe that the intrinsic numerical range is a compact and convex subset of $\mathbb{K}$.

The second numerical range we will deal with is the spatial numerical range, which extends the corresponding definition for bounded linear operators.

Definition 1.2 *(Spatial numerical range).* Let Y be a Banach space and let Γ be a non-empty set. We fix $g \in \ell_\infty(\Gamma, Y)$ with $\|g\| = 1$. For every $f \in \ell_\infty(\Gamma, Y)$, the *spatial numerical range* of f relative to g is given by

$$W_g(f) := \{y^*(f(t)) \,:\, y^* \in S_{Y^*},\ t \in \Gamma,\ y^*(g(t)) = 1\}.$$