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Singular values of convex functions of operators and the arithmetic–geometric mean inequality

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A R T I C L E I N F O

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A B S T R A C T

We prove singular value inequalities for convex functions of products and sums of operators that generalize the arithmetic–geometric mean inequality for operators. Among other results, we prove that if A_i, B_i, X_i, Y_i, $i = 1, \ldots, n$ are operators on a complex separable Hilbert space such that $\frac{|X_i|^2+|Y_i|^2}{2n} \leq I$, $i = 1, \ldots, n$ and if f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$, then

$$s_j \left(f \left(\left| \sum_{i=1}^{n} A_i X_i Y_i^* B_i^* \right| \right) \right) \leq \frac{1}{2} s_j \left(\bigoplus_{k=1}^{n} \left(X_k^* \left(\sum_{i=1}^{n} f\left(|A_i^* A_k|\right) \right) X_k + Y_k^* \left(\sum_{i=1}^{n} f\left(|B_i^* B_k|\right) \right) Y_k \right) \right)$$

for $j = 1, 2, \ldots$.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space $\mathbb{H}$. For a compact operator $A \in \mathbb{B}(\mathbb{H})$, let $s_1(A), s_2(A), \ldots$ denote the singular values of A (i.e. the eigenvalues of $|A| = (A^*A)^{1/2}$) arranged in decreasing order and repeated according to multiplicity. For the sake of brevity, when we consider $s_j(A)$, we are assuming that A is a compact operator. Some of the important properties of singular values are that

$$s_j(A^*A) = s_j(AA^*) \tag{1.1}$$

and the singular values of operators are unitarily invariance, that is

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$$s_j(UAV) = s_j(A) \tag{1.2}$$

for $j = 1, 2, \ldots$, where U, V are any unitary operators in $\mathbb{B}(\mathbb{H})$.

By the symbol $||\cdot||$ we mean the usual operator norm, which is defined on all of $\mathbb{B}(\mathbb{H})$. In fact, if $A, X, Y \in \mathbb{B}(\mathbb{H})$ such that A is compact, then $||A|| = s_1(A)$ and

$$s_j(XAY) \leq \|X\| \, \|Y\| \, s_j\,(A) \tag{1.3}$$

for $j = 1, 2, \ldots$ (see, e.g., [2, p. 75] or [5, p. 27]).

The direct sum of operators $A_1, \ldots, A_n$ in $\mathbb{B}(\mathbb{H})$, denoted by $\bigoplus_{k=1}^{n} A_k$, is the block-diagonal operator matrix defined on $\bigoplus_{k=1}^{n} \mathbb{H}$ by

$$\bigoplus_{k=1}^{n} A_k = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{bmatrix}.$$

For $n = 2$, we write $A_1 \oplus A_2$ instead of $\bigoplus_{k=1}^{2} A_k$.

It is clear that for $A, B \in \mathbb{B}(\mathbb{H})$,

$$s_j(A) \leq s_j(B), j = 1, 2, \ldots \Leftrightarrow s_j(A \oplus 0) \leq s_j(B \oplus 0), j = 1, 2, \ldots, \tag{1.4}$$

$$s_j(A) \leq s_j(B), j = 1, 2, \ldots \Leftrightarrow s_j(A \oplus 0) \leq s_j(B), j = 1, 2, \ldots, \tag{1.5}$$

and

$$s_j(A) \leq s_j(B), j = 1, 2, \ldots \Leftrightarrow s_j(A) \leq s_j(B \oplus 0), j = 1, 2, \ldots \tag{1.6}$$

The arithmetic–geometric mean inequality for scalars asserts that if $a, b \in \mathbb{C}$, then

$$|ab| \leq \frac{|a|^2 + |b|^2}{2}.$$

An elegant generalization of this inequality has been given in [3]. This generalization asserts that if $A, B \in \mathbb{B}(\mathbb{H})$, then

$$s_j(AB^*) \leq \frac{1}{2} s_j\left(|A|^2 + |B|^2\right) \tag{1.7}$$

for $j = 1, 2, \ldots$. The inequality (1.7) has attracted several mathematicians and a lot of papers have been written on this subject.

Recently, in [1] a new generalization of the inequality (1.7) has been given. This generalization asserts that if $A, B, X \in \mathbb{B}(\mathbb{H})$, then

$$s_j(AXB^*) \leq \frac{\|X\|}{2} s_j\left(|A|^2 + |B|^2\right) \tag{1.8}$$

for $j = 1, 2, \ldots$. Moreover, it is shown in [1] that if $A_i, B_i, X_i \in \mathbb{B}(\mathbb{H})$, $i = 1, \ldots, n$, such that X_i is positive, $i = 1, \ldots, n$, then