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Measure zero stability problem of a new quadratic functional equation

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ABSTRACT

Let $\mathbb{R}$ be the set of real numbers, Y a Banach space and $f : \mathbb{R} \to Y$. We prove the Ulam–Hyers stability theorem for the new quadratic functional equation

$$\sum_{j=2}^{k} \big(f(x_1 + x_j) + f(x_1 - x_j)\big) = 2(k-1)f(x_1) + 2\sum_{j=2}^{k} f(x_j)$$

for all $(x_1, \ldots, x_k) \in \Gamma$, where $\Gamma \subset \mathbb{R}^k$ is of Lebesgue measure 0. Using the result we obtain an asymptotic behavior of the equation.

1. Introduction

Throughout the paper, we denote by X and Y a real normed space and a real Banach space, respectively and $\epsilon \geq 0$. A mapping $f : X \to Y$ is quadratic if f satisfies the equation

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0 \tag{1.1}$$

for all $x, y \in X$. F. Skof was the first author who proved the Ulam–Hyers stability of the quadratic functional equation [22].

Theorem 1.1. *Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

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$$\|f(x) - Q(x)\| \leq \frac{1}{2}\epsilon$$

for all $x \in X$.

It is a very natural subject to consider functional equations or inequalities satisfied on restricted domains or satisfied under restricted conditions (see [1–6,8,9,11,14,15,18,20,21] for related results). In particular we refer to [7] for most recent developments on conditional stability for functional equations. Among the results, S.M. Jung and J.M. Rassias proved the Ulam–Hyers stability of the quadratic functional equations in a restricted domain [13,17].

Theorem 1.2. *Let* $d > 0$. *Suppose that* $f : X \to Y$ *satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon \tag{1.2}$$

for all $x, y \in X$ *with* $\|x\| + \|y\| \geq d$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{7}{2}\epsilon \tag{1.3}$$

for all $x \in X$.

It is very natural to ask if the restricted domain $D := \{(x, y) \in X^2 : \|x\| + \|y\| \geq d\}$ can be replaced by a much smaller subset $\Gamma \subset D$ (e.g. a subset of measure 0 in a measure space X). In the paper [10], the stability of (1.2) was considered in a set $\Gamma \subset \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of measure $m(\Gamma) = 0$ when $f : \mathbb{R} \to Y$. As a result it is proved that if $f : \mathbb{R} \to Y$ satisfies (1.2) for all $(x, y) \in \Gamma$, then there exists a unique quadratic function $Q : \mathbb{R} \to Y$ satisfying (1.3). As a consequence, it is also proved that if f satisfies the asymptotic condition

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \to 0 \tag{1.4}$$

as $|x| + |y| \to \infty$ in Γ, then f is quadratic.

In Section 2 of the paper, as an abstract approach we consider the Ulam–Hyers stability of a new quadratic functional equation (see [19])

$$\left\| \sum_{j=2}^{k} \big(f(x_1 + x_j) + f(x_1 - x_j)\big) - 2(k-1)f(x_1) - 2\sum_{j=2}^{k} f(x_j) \right\| \leq \epsilon \tag{1.5}$$

for all $(x_1, \ldots, x_k) \in \Omega$, where $\Omega \subset X^k$ satisfies the condition denoted by (C).

In Section 3 of the paper, based on the Baire category theorem we construct a subset $\Gamma \subset \mathbb{R}^k$ of Lebesgue measure $m(\Gamma) = 0$ which satisfies the condition (C). As a result we prove that if $f : \mathbb{R} \to Y$ satisfies (1.5) for all $(x_1, \ldots, x_k) \in \Gamma$, then there exists a unique quadratic function $Q : \mathbb{R} \to Y$ such that

$$\left\| f(x) - Q(x) - \frac{f(0)}{3} \right\| \leq \frac{7\epsilon}{3(k-1)}$$

for all $x \in \mathbb{R}$. As a consequence of our result we also prove that $f : \mathbb{R} \to Y$ satisfies the asymptotic condition

$$\left\| \sum_{j=2}^{k} \big(f(x_1 + x_j) + f(x_1 - x_j)\big) - 2(k-1)f(x_1) - 2\sum_{j=2}^{k} f(x_j) \right\| \to 0$$

as $|x_1| + |x_2| + \ldots + |x_k| \to \infty$ in Γ if and only if f is quadratic.