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Cross-sections of solution funnels ☆

Petr Hájek [a,*], Paola Vivi [b]

[a] Mathematical Institute, Czech Academy of Science, Žitná 25, 115 67 Praha 1, Czech Republic
[b] Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Zikova 4, 166 27, Prague, Czech Republic

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ABSTRACT

Let X be a separable infinite dimensional real Banach space. We denote by $\mathcal{F}(X)$ the class of continuous functions $f : \mathbb{R} \times X \to X$ such that the ODE $u' = f(t,u), u(t_0) = x,\ t_0 \in \mathbb{R}, x \in X$, has a global solution for any initial condition. Our main result states that $A \subset X$ is the cross-section of a solution funnel of the ODE $u' = f(t,u), u(0) = 0$, for some $f \in \mathcal{F}(X)$, if and only if A is an analytic set.

1. Introduction

The study of cross-sections of solution funnels forms a classical subject in ODE ever since the pioneering work of Peano, Hukuhara, Aronszajn, and Kneser. The problem of characterizing which sets in X are the cross-sections of a solution funnel, for a suitable continuous mapping f, was thoroughly investigated in the finite dimensional spaces X in [13]. Pugh characterized the cross-section sets in the special case when the initial condition is the only non-uniqueness initial condition, leaving the general problem open.

The main result of the present note is a complete characterization of the cross-sections of the solution funnels of ODE in separable infinite dimensional Banach spaces as analytic sets.

Let us start by giving a more detailed exposition of the main concepts and results in this area. Let X be a real Banach space and $f : \mathbb{R} \times X \to X$ be a continuous mapping. We are concerned with the initial value problem

$$u' = f(t,u), \tag{1}$$

$$u(t_0) = x, \quad t_0 \in \mathbb{R}, x \in X. \tag{2}$$

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* Corresponding author.
E-mail addresses: hajek@math.cas.cz (P. Hájek), vivi@math.feld.cvut.cz (P. Vivi).

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Given an open interval $J \subset \mathbb{R}$, we say that $u : J \to X$ is a solution to (1) if u is a differentiable function and $u'(t) = f(t, u), t \in J$. If $t_0 \in J$ and (2) holds, we say that the solution satisfies the initial condition. If $J = \mathbb{R}$, we say that the solution is global.

If X is finite dimensional, then the system (1), (2) has a local solution by the classical Peano theorem. Every solution can be extended to a maximal solution, which is not necessarily a global solution. Moreover, for a given initial condition, the solution may not be unique. In fact there are examples failing the uniqueness for any initial condition (first by Lavrentieff see [7]). On the other hand, given an initial condition $u(t_0) = x$ there exists an open interval $(a, b) = I \ni t_0$ such that every solution of (1), (2) is extendable to the whole I. It is therefore natural to study the solution sets which share the same initial condition and the same domain.

Definition 1.1. Let $(t_0, x) \in \mathbb{R} \times X$, $t_0 \in I$ where I is an interval. Suppose that every solution to (1), (2) is extendable to I. The overlying solution funnel from (t_0, x) on I is the set of all solutions on I, i.e.

$$F(f, (t_0, x)) = \{u : I \to X : u(t_0) = x, u'(t) = f(t, u) \text{ on } I\} \tag{3}$$

Given $t \in I$, we say that $S_t(f, (t_0, x)) = \{u(t) : \text{ where } u \in F(f, (t_0, x))\}$ is a cross-section of the solution funnel from (t_0, x) at time t. For completeness, let us recall that the solution funnel is the set

$$S(f, (t_0, x)) = \{(t, u(t)) \in I \times X : u(t) \in S_t(f, (t_0, x))\} \tag{4}$$

In order to facilitate the study of solution funnels (namely the problem of the domain of a particular solution) the following class of functions has been introduced. $\mathcal{F}^m$ is the set of all continuous $f : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ which have supports in sets of the form $\mathbb{R} \times$ *compact*. If $f \in \mathcal{F}^m$ then every solution can be extended into a global solution. With this definition Kneser [11] proved that if $f \in \mathcal{F}^m$ then every cross-section of the solution funnel of f is a continuum (a topological space is called continuum if it is nonempty, compact, and connected). Hukuhara [9] proved a stronger result that the overlying solution funnel is a continuum (Aronszajn [1] further improved this result). Pugh [13] and Horst [8] gave examples showing that the overlying funnel need not be arcwise connected.

A detailed study of finer properties of cross-sections of solution funnels was carried out by Pugh [13], and Rogers [14]. Pugh constructed continua in $\mathbb{R}^n, n \geq 2$ that are not cross-sections of the solution funnels (a spiral tending to a circle), and cross-sections that are not arcwise connected. Every C^1-smooth polyhedron is a cross-section of the solution funnel.

In spite of the wealth of information contained in [13], a characterization of cross-sections of solution funnels in finite-dimensional spaces remains open.

Studying the solution funnels in infinite-dimensional spaces again involves the question of the existence and the domain of solutions. (Some examples do not pay attention to the global behaviour of the equation, but it is natural to consider functions that behave well with respect to the existence. Note that the local existence problem has been solved in [15] and [6] in the negative for all separable Banach spaces.)

Definition 1.2. Let X be a real Banach space. Denote by $\mathcal{F}(X)$ the class of continuous functions $f : \mathbb{R} \times X \to X$ such that (1) has a solution for every initial condition, that extends to a global solution.

It is well-known that a simple condition that guarantees the existence and uniqueness of the local solution is the Lipschitz condition on f in the second variable. We are going to use this fact in our construction. Binding [3] proved that for every infinite dimensional Banach space X there is a function $f \in \mathcal{F}(X)$ such that the closed unit ball B_X is a cross-section of the solution funnel. Using ideas of Godunov, Binding constructs non-connected funnel cross-sections (but $f \notin \mathcal{F}(X)$). Deimling [4, p. 24] proves that the overlying solution funnel is a continuum, provided the bounded and uniformly continuous function f preserves the Kuratowski measure of non-compactness for subsets of X. Further progress in the study of funnels comes from the