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On a class of generalized Takagi functions with linear pathwise quadratic variation

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ABSTRACT

We consider a class $\mathscr{X}$ of continuous functions on $[0,1]$ that is of interest from two different perspectives. First, it is closely related to sets of functions that have been studied as generalizations of the Takagi function. Second, each function in $\mathscr{X}$ admits a linear pathwise quadratic variation and can thus serve as an integrator in Föllmer's pathwise Itō calculus. We derive several uniform properties of the class $\mathscr{X}$. For instance, we compute the overall pointwise maximum, the uniform maximal oscillation, and the exact uniform modulus of continuity for all functions in $\mathscr{X}$. Furthermore, we give an example of a pair $x, y \in \mathscr{X}$ for which the quadratic variation of the sum $x + y$ does not exist.

1. Introduction

In this note, we study a class $\mathscr{X}$ of continuous functions on $[0,1]$ that is of interest from several different perspectives. On the one hand, just as typical Brownian sample paths, each function $x \in \mathscr{X}$ admits the linear pathwise quadratic variation, $\langle x\rangle_t = t$, in the sense of Föllmer [13] and therefore can serve as an integrator in Föllmer's pathwise Itō calculus. On the other hand, $\mathscr{X}$ is a subset, or has a nonempty intersection, with classes of functions that have been studied as generalizations of Takagi's celebrated example [29] of a nowhere differentiable continuous function. We will now explain the connections of our results with these two separate strands of literature.

1.1. *Contributions to Föllmer's pathwise Itō calculus*

In 1981, Föllmer [13] proposed a pathwise version of Itō's formula, which, as a consequence, yields a strictly pathwise definition of the Itō integral as a limit of Riemann sums. Some recent developments have

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led to a renewed interest in this pathwise approach. Among these is the conception of *functional* pathwise Itō calculus by Dupire [10] and Cont and Fournié [6,7], which for instance is crucial in defining partial differential equations on path space [11]. Another source for the renewed interest in pathwise Itō calculus stems from the growing awareness of model ambiguity in mathematical finance and the resulting desire to reduce the reliance on probabilistic models; see, e.g., [15] for a recent survey and [3,4,8,14,26,27] for case studies with successful applications of pathwise Itō calculus to financial problems. A systematic introduction to pathwise Itō calculus, including an English translation of [13], is provided in [28].

A function $x \in C[0,1]$ can serve as an integrator in Föllmer's pathwise Itō calculus if it admits a continuous pathwise quadratic variation $t \mapsto \langle x\rangle_t$ along a given refining sequence of partitions of $[0,1]$. This condition is satisfied whenever x is a sample path of a continuous semimartingale, such as Brownian motion, and does not belong to a certain nullset. This nullset, however, is generally not known explicitly, and so it is not possible to tell whether a specific realization x of Brownian motion does indeed admit a continuous pathwise quadratic variation. The first purpose of this note is to provide a rich class $\mathscr{X}$ of continuous functions that can be constructed in a straightforward manner and that do admit the nontrivial pathwise quadratic variation $\langle x\rangle_t = t$ for all $x \in \mathscr{X}$. The functions in $\mathscr{X}$ can thus be used as a class of test integrators in pathwise Itō calculus. Our corresponding result, Proposition 2.6, slightly extends a previous result by Gantert [18,19], from which it follows that $\langle x\rangle_1 = 1$ for all $x \in \mathscr{X}$.

Still within this context, a second purpose of this note is to investigate whether the existence of $\langle x\rangle$ and $\langle y\rangle$ implies the existence of $\langle x+y\rangle$ (or, equivalently, the existence of the pathwise quadratic covariation $\langle x, y\rangle$). For typical sample paths of a continuous semimartingale, this implication is always true, but the corresponding nullset will depend on both x and y. In the literature on pathwise Itō calculus, however, it has been taken for granted that the existence of $\langle x+y\rangle$ cannot be deduced from the existence of $\langle x\rangle$ and $\langle y\rangle$. In Proposition 2.7 we will now give an example of two functions $x, y \in \mathscr{X}$ for which $\langle x+y\rangle$ does indeed not exist. To the knowledge of the author, such an example has so far been missing from the literature.

1.2. Contributions to the theory of generalized Takagi functions

In 1903, Takagi [29] proposed an example of a continuous function on $[0,1]$ that is nowhere differentiable. This function has since been rediscovered several times and its properties have been studied extensively; see the recent surveys by Allaart and Kawamura [2] and Lagarias [24]. While the original Takagi function itself does not belong to our class $\mathscr{X}$, there are at least two classes of functions whose study was motivated by the Takagi function and that are intimately connected with $\mathscr{X}$. One family of functions is the "Takagi class" introduced in 1984 by Hata and Yamaguti [20]. Similar but more restrictive function classes were introduced earlier by Faber [12] or Kahane [21]. The Takagi class has a nonempty intersection with $\mathscr{X}$ but neither one is included in the other. More recently, Allaart [1] extended the Takagi class to a more flexible class of functions. This family now contains $\mathscr{X}$. By extending arguments given by Kôno [23] for the Takagi class, Allaart [1] studies in particular the moduli of continuity of certain functions in his class.

In contrast to these previous studies, the focus of this paper is not so much on the individual features of functions $x \in \mathscr{X}$ but rather on uniform properties of the entire class $\mathscr{X}$. Here we compute the overall pointwise maximum, the uniform maximal oscillation, and the exact uniform modulus of continuity for all functions in $\mathscr{X}$. In these computations, we cannot use previous methods that were conceived for the analysis of the Takagi functions and its generalizations. For instance, neither the result and arguments from Kôno [23] nor the ones from Allaart [1] apply to the modulus of continuity of functions in $\mathscr{X}$, and a suitable extension of the previous approaches must be developed. This new extension exploits the self-similar structure of $\mathscr{X}$ and its members.

A special role in our analysis will be played by the function $\widehat{x}$, defined in (2.2) below. It has previously appeared in the work of Ledrappier [25], who studied the Hausdorff dimension of its graph, and in Gantert [18,19]. Here we will determine its global maximum and its exact modulus of continuity. In particular