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Generalized eigenspaces of generators of evolution semigroups

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ABSTRACT

A periodic evolutionary process $\{U(t,s)\}_{t\geq s}$ with period τ on a Banach space X gives an evolution semigroup with the generator L on the space of τ-periodic continuous X-valued functions. We prove that the generalized eigenspaces $N_L = \mathcal{N}((\alpha I - L)^n)$ and $N_U = \mathcal{N}((e^{\alpha\tau} I - U(\tau,0))^n)$ are isomorphic. The algebraic representation of a function $u \in N_L$ is given by the initial value $u(0) = w \in N_U$.

1. Introduction

Let $\{U(t,s)\}_{t\geq s}$ be a periodic evolutionary process with period $\tau > 0$ on a complex Banach space X. Denote by L the generator of the evolution semigroup $\{T^h\}_{h\geq 0}$, associated with $\{U(t,s)\}_{t\geq s}$, on the phase space $P_\tau(X)$, the space of all τ-periodic continuous X-valued functions. In [7], we investigated the existence of periodic solutions to the equation $Lu = \epsilon F(u,\epsilon)$, $u \in P_\tau(X)$, $\epsilon > 0$, where F is a Nemitskii operator. This equation is an abstract form of a nonlinear periodic differential equation. The essential points in the proof of the existence of periodic solutions are the structure of the eigenspace of L and the relationship between normal eigenvalues of the generator L and those of the monodromy operator $V(0) := U(\tau, 0)$. However, these points were only partially stated in connection with the existence of periodic solutions. For example, we showed that $\dim \mathcal{N}(L^n) \leq \dim \mathcal{N}((I - V(0))^n)$, where $\mathcal{N}(H)$ stands for the null space of a linear operator H, but we did not discuss whether or not the equality holds.

The aim of this paper is to show that, for any complex number α and any integer $n \geq 1$, the following isomorphic relation holds:

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$$\mathcal{N}((\alpha I - L)^n) \cong \mathcal{N}((e^{\alpha\tau} I - V(0))^n). \tag{1}$$

Hence,

$$\dim \mathcal{N}((\alpha I - L)^n) = \dim \mathcal{N}((e^{\alpha\tau} I - V(0))^n) \tag{2}$$

in place of the inequality stated in [7, Lemma 4.5]. This equality implies that the normal eigenvalue α of L and the normal eigenvalue $e^{\alpha\tau}$ of $V(0)$ have the same index (see [5]). We adopt the definition of normal eigenvalues given in [2,12].

To show (1), we will give a representation of solutions to the equation

$$(\alpha I - L)^n u = 0 \tag{3}$$

for $u \in \mathcal{D}(L) \subset P_\tau(x)$. It will be difficult to do so if we consider the general phase spaces for $\{T^h\}_{h\geq 0}$. Instead, we select the minimum space $P_\tau(X)$ for the phase space of $\{T^h\}_{h\geq 0}$. Then, we obtain a value $u(t)$, $t \geq 0$, for the solution $u \in \mathcal{D}(L^n) \subset P_\tau(X)$ to Eq. (3) in the form

$$u(t) = e^{-\alpha t} U(t,0) \sum_{m=0}^{n-1} \frac{t^m}{m!} w_m, \quad t \geq 0, \tag{4}$$

where $w_0 \in \mathcal{N}((e^{\alpha\tau} I - V(0))^n)$, and the subsequent $w_1, w_2, \cdots, w_{n-1}$ are given by w_0 (see Lemma 4.3). This form is rewritten as

$$u(t) = e^{-\alpha t} U(t,0) \sum_{k=0}^{n-1} \left(\frac{t}{\tau}\right)_{\bar{k}} \frac{1}{\mu^k k!} (\mu I - U(\tau,0))^k w_0,$$

where $\mu = e^{\alpha\tau}$, using polynomials

$$(t)_{\bar{k}} = t(t+1)(t+2)\cdots(t+k-1), \quad k \geq 1, \quad (t)_{\bar{0}} = 1. \tag{5}$$

Equalities (1) and (2) follow from this main result on the algebraic structure of $u(t)$ (see Theorem 2).

The present paper is organized as follows. In Section 2, we introduce the evolution semigroup together with its generator L on $P_\tau(X)$ associated with $U(t,s)$. The solution u of the equation $(\alpha I - L)u = g$ is given together with the condition for the initial value $u(0) = w$. In Section 3, the equation $(\alpha I - L)^n u = 0$ is transformed to the equivalent system of first-order equations. Solving this system, we obtain the system of equations for the coefficients $w_0, w_1, \cdots, w_{n-1}$ in (4). In Section 4, this system is solved with respect to $w_1, \cdots, w_{n-1}$ using $w_0 \in \mathcal{N}((e^{\alpha\tau} I - V(0))^n)$ and Stirling numbers. It is a by-product of the congruence (1) that the semigroup $\{T^h\}_{h\geq 0}$ is not an eventually compact semigroup provided that $V(0)$ has a nonzero eigenvalue (Corollary 4.5). We will prove other spectral properties in evolutionary processes in forthcoming papers [5,8].

2. Preliminaries

Let X be a complex Banach space equipped with a norm $\|\cdot\|$, and $P_\tau(X)$ be the space of all periodic continuous X-valued functions with period $\tau > 0$. Define the norm of $g \in P_\tau(X)$ by $\|g\| = \max_{0\leq t\leq\tau} \|g(t)\|$. A family of bounded linear operators $\{U(t,s)\}_{t\geq s}$, $(t, s \in \mathbb{R})$ on a Banach space X to itself is called *a τ-periodic* (*strongly continuous*) *evolutionary process* if the following conditions are satisfied: