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Well-posedness and exponential decay of solutions for the Blackstock–Crighton–Kuznetsov equation ☆

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A B S T R A C T

The present work provides well-posedness and exponential decay results for the Blackstock–Crighton–Kuznetsov equation arising in the modeling of nonlinear acoustic wave propagation in thermally relaxing viscous fluids. First, we treat the associated linear equation by means of operator semigroups. Moreover, we derive energy estimates which we will use in a fixed-point argument in order to obtain well-posedness of the Blackstock–Crighton–Kuznetsov equation. Using a classical barrier argument we prove exponential decay of solutions.

1. Introduction

The present work aims to enhance the mathematical understanding of nonlinear acoustic wave propagation in viscous, thermally conducting, inert fluids. In particular, our motivation is to deal with higher order models arising in nonlinear acoustics. An acoustic wave propagates through a medium as a local pressure change. Nonlinear phenomena typically occur at high acoustic pressures which are used for several medical and industrial purposes such as lithotripsy, thermotherapy, ultrasound cleaning and sonochemistry. Due to this broad range of applications, nonlinear acoustics is currently an active field of research, see [3,4,8,9,11–16,10,17,19–21] and the references therein.

The classical models in nonlinear acoustics are partial differential equations of second order in time which are characterized by the presence of a viscoelastic damping. The most general of these conventional models is Kuznetsov's equation

$$u_{tt} - c^2\Delta u - b\Delta u_t = \left(\tfrac{1}{c^2}\tfrac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_t, \tag{1.1}$$

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where u denotes the acoustic velocity potential, $c > 0$ is the speed of sound, $b \geq 0$ is the diffusivity of sound and B/A is the parameter of nonlinearity. Neglecting local nonlinear effects (in the sense that the expression $c^2|\nabla u|^2 - (u_t)^2$ is sufficiently small) one arrives at the Westervelt equation

$$u_{tt} - b\Delta u_t - c^2\Delta u = \left(\tfrac{1}{c^2}\left(1 + \tfrac{B}{2A}\right)(u_t)^2\right)_t. \tag{1.2}$$

Both, the Kuznetsov and the Westervelt equation, can alternatively be formulated in terms of the acoustic pressure p via the relation $\rho u_t = p$, where ρ denotes the mass density. The quantities A and B occurring in the parameter of nonlinearity are the coefficients of the first and second order terms in the Taylor series expansion of the variation of pressure in the medium in terms of variation of the density. For a detailed introduction to the field of nonlinear acoustics we refer to [8].

The Kuznetsov equation in its turn can be regarded as a simplification (for a small thermal conductivity $a = \nu \mathrm{Pr}^{-1}$, where ν is the kinematic viscosity and Pr is the Prandtl number) of the higher order model

$$(a\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left(\tfrac{1}{c^2}\tfrac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_{tt} \tag{1.3}$$

which we call Blackstock–Crighton–Kuznetsov equation. Neglecting local nonlinear effects as it is done when reducing the Kuznetsov equation to the Westervelt equation we arrive at the Blackstock–Crighton–Westervelt equation

$$(a\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left(\tfrac{1}{c^2}\left(1 + \tfrac{B}{2A}\right)(u_t)^2\right)_{tt}. \tag{1.4}$$

For more information on the derivation of (1.3) and (1.4) we refer to Section 2.

The Westervelt and the Kuznetsov equation as well as the Khoklov–Zabolotskaya–Kuznetsov equation, which is another standard model in nonlinear acoustics, have recently been quite extensively investigated (see, e.g., [4,9,11,13,12,14,16,19]). Research on higher order models such as (1.3) and (1.4) is still in an early stage. The starting point was [3] where well-posedness and exponential decay of solutions for (1.4) together with homogeneous Dirichlet boundary conditions was shown. The goal of the present paper is to provide results on well-posedness and exponential decay for the more general Blackstock–Crighton–Kuznetsov equation (1.3) which is one more step towards closing the gap of missing results on higher order models in nonlinear acoustics.

More precisely, the present work is devoted to the homogeneous Dirichlet boundary value problem

$$\begin{cases} (a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = (k(u_t)^2 + s|\nabla u|^2)_{tt} & \text{in } (0,T)\times\Omega, \\ (u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t=0\}\times\Omega, \\ (u, \Delta u) = (0,0) & \text{on } [0,T)\times\Gamma, \end{cases} \tag{1.5}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ of dimension $n \in \{1,2,3\}$ with smooth boundary $\Gamma = \partial\Omega$, where $T > 0$ is either finite or infinite. The initial values $u_0, u_1, u_2 \colon \Omega \to \mathbb{R}$ are given and $u\colon [0,T)\times\Omega \to \mathbb{R}$ is the unknown. Moreover, we assume that $a, b, c, k > 0$ are constants and $s \in \{0,1\}$. The case $s = 1$ corresponds to (1.3) whereas $s = 0$ relates to (1.4). The restriction on the dimension of the spatial domain Ω is imposed in order to be able to use the embedding $H^2(\Omega) \hookrightarrow L_\infty(\Omega)$ which we will do at several crucial steps. We therefore point out that our results do not hold for $n \geq 4$. However, this is not of relevance in practical applications anyway.

Here, beside the classical Dirichlet boundary condition $u|_\Gamma = 0$ we impose $\Delta u|_\Gamma = 0$, since this ensures that both, $a\Delta u - u_t$ and $u_{tt} - b\Delta u_t - c^2\Delta u$, have homogeneous Dirichlet boundary conditions such that the homogeneous Dirichlet Laplacian can be applied. In particular, $\Delta u|_\Gamma = 0$ allows us to interchange the differential expressions on the left-hand side which we will do when deriving energy estimates.