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The matching of two stable sewing linear systems in the plane

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A B S T R A C T

In this article we study one of the main problems in the qualitative theory of planar differential equations: the problem of determining the basin of attraction of an equilibrium point. We give a rigorous proof that for planar sewing piecewise linear systems with two zones, defined by Hurwitz matrices the unique equilibrium point in the separation straight line is globally asymptotically stable. On the other hand, we prove that sewing piecewise linear systems with two zones in the plane, defined by Hurwitz matrices can have one unstable equilibrium point at the origin allowing a broken line to separate the zones, leading to counter-intuitive dynamical behaviors of simple piecewise linear systems in the plane.

1. Introduction and statement of the main results

Piecewise linear differential systems with two zones in the plane are generally defined by

$$X' = \begin{cases} A^- X + B^-, \ \mathcal{H}(X) \le 0, \\ A^+ X + B^+, \ \mathcal{H}(X) \ge 0, \end{cases} \tag{1}$$

where the prime denotes derivative with respect to the independent variable t, called here the time, $X = (x, y) \in \mathbb{R}^2$, $A^{\pm}$ are 2×2 real matrices, $B^{\pm}$ are 2×1 real matrices, the function $\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}$ is at least continuous and the set $\Sigma = \mathcal{H}^{-1}(0)$ divides the plane in two unbounded components (zones) Σ^+ and Σ^- where $\mathcal{H}$ is positive and negative, respectively. Thus $\mathbb{R}^2 = \Sigma^+ \cup \Sigma \cup \Sigma^-$.

Since the seminal work of Andronov et al. [1], a lot of articles were published mainly about questions like the existence, number, stability and distribution of limit cycles. See [2–4,7,9–13] and the references therein. These studies were developed taking into account aspects like the number and stability of equilibrium points as well as their locations with respect to the separation boundary Σ and the behavior of the linear vector

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fields on Σ. Usually, the points of discontinuity on the separation boundary are classified as sewing, sliding, escaping or tangency points [13]. Recall that a point $X_0 = (x_0, y_0) \in \Sigma = \mathcal{H}^{-1}(0)$ is a sewing (crossing) point if

$$\left(\left(A^- X_0 + B^-\right) \cdot \nabla \mathcal{H}(X_0)\right) \left(\left(A^+ X_0 + B^+\right) \cdot \nabla \mathcal{H}(X_0)\right) > 0.$$

Nevertheless, to the best of our knowledge there are few articles studying the problem of the global stability of an equilibrium point in this class of systems (see [8] for a study and application in control theory). The landmark of such a study is the article [6] in which the authors analyzed the case where:

H1. A^+ and A^- are Hurwitz matrices (the real parts of all its eigenvalues are negative);
H2. $B^+ = B^- = 0$;
H3. the separation boundary Σ is a straight line that contains the unique equilibrium point at the origin;
H4. the vector fields A^+X and A^-X are continuous on $\Sigma - \{(0,0)\}$.

With the above hypotheses, in [6] was proved that the unique equilibrium point is globally asymptotically stable.

In this article we extend the above result changing the hypothesis **H4** to the following one

H4'. the points on $\Sigma - \{(0,0)\}$ are of sewing (or crossing) type.

More precisely, in Section 2 we prove the following theorem.

Theorem 1. *Consider system (1) with the hypotheses* ***H1****,* ***H2****,* ***H3*** *and* ***H4'****. Then the unique equilibrium point at the origin is globally asymptotically stable.*

It is appropriate to remark that the hypotheses in Theorem 1 assure that system (1) is a well-posed planar bimodal system in the terminology of control theory (see [8]).

The separation boundary Σ between the two zones plays an important role in planar discontinuous piecewise linear differential systems. In the article [3] was exhibited an example of such a system with seven limit cycles having Σ as a broken line. In the article [4] the authors proved the existence of a class of discontinuous piecewise linear differential systems with two zones in the plane separated by a broken line Σ having exactly n hyperbolic limit cycles, for any given positive integer n.

In Section 3 we prove the following theorem where instead of **H3** we consider the hypothesis:

H3'. the separation boundary Σ is a broken line that contains the unique equilibrium point at the origin.

Theorem 2. *Consider system (1) with the hypotheses* ***H1****,* ***H2****,* ***H3'*** *and* ***H4'****. There are Hurwitz matrices A^+ and A^- such that the unique equilibrium point at the origin is either a stable focus, or a center, or an unstable focus.*

This article is also related to [5] where the authors proved that continuous piecewise linear systems in $\mathbb{R}^3$ with two zones separated by a plane and defined by Hurwitz matrices can have an unstable equilibrium point at the origin.

Finally, in Section 4, we make some concluding remarks.