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Existence, uniqueness and conditional stability of periodic solutions to evolution equations

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A B S T R A C T

Using an ergodic approach, we investigate the condition for existence and uniqueness of periodic solutions to linear evolution equation $\dot{u} = A(t)u + f(t)$, $t \geq 0$, and to semi-linear evolution equations of the form $\dot{u} = A(t)u + g(u)(t)$, where the operator-valued function $t \mapsto A(t)$ and the vector-valued function $f(t)$ are T-periodic, and Nemytskii's operator g is locally Lipschitz and maps T-periodic functions to T-periodic functions. We then apply the results to study the existence, uniqueness, and conditional stability of periodic solutions to the above semi-linear equation in the case that the family $(A(t))_{t\geq 0}$ generates an evolution family having an exponential dichotomy.

1. Introduction

Consider the abstract semi-linear evolution equation

$$\dot{u} = A(t)u + g(u)(t), \quad t \in \mathbb{R}_+, \tag{1.1}$$

where for each $t \in \mathbb{R}_+$, $A(t)$ is a possibly unbounded operator on a Banach space X such that $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on X, and the so-called Nemytskii's operator g is locally Lipschitz and acts on some function space of vector-valued functions such as $C_b(\mathbb{R}_+, X)$. One of the important research directions related to the asymptotic behavior of the solutions to the above equation is to find conditions for the existence of a periodic solution to that equation in case g maps T-periodic functions to T-periodic functions. Beside some approaches which seem to be suitable only for specific equations like Tikhonov's fixed point method [12] or the Lyapunov functionals [14], the most popular approaches for proving the

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existence of a periodic solution are the ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings (see [1,6,11–14] and the references therein).

However, in some concrete applications, e.g., to partial differential equations in unbounded domains or to equations that have unbounded solutions, such compact embeddings are no longer valid, and the existence of bounded solutions is not easy to obtain since one has to carefully choose an appropriate initial vector (or conditions) to guarantee the boundedness of the solution emanating from that vector.

Therefore, in the present paper, we propose another approach toward the existence and uniqueness of the periodic solution to the abstract evolution equation (1.1), so that it can help to overcome such difficulties. Namely, we start with the linear equation

$$\dot{u} = A(t)u + f(t), \quad t \geq 0 \tag{1.2}$$

and use an ergodic approach (see [15] for the origin of the approach) to prove the existence of a periodic solution through the existence of a bounded solution whose sup-norm can be controlled by the sup-norm of the input function f. We refer the reader to [5] for an extension of such an approach to the case of periodic solutions to Stokes and Navier–Stokes equations around rotating obstacles.

We then use the fixed point argument to prove such results for the abstract semi-linear evolution equation (1.1). Our approach invokes somehow the folklore methodology of Massera [7] for periodic solutions to Ordinary Differential Equations (which roughly said that if an ODE has a bounded solution then it has a periodic one) to the level of general Banach spaces.

It is worth noting that our framework fits perfectly to the situation of exponentially dichotomic linear parts, i.e., the case when family $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ having an exponential dichotomy (see Definition 4.1 below), since in this case we can choose the initial vector from that emanates a bounded solution. We can also prove the conditional stability of periodic solutions in this case.

Our main results are contained in Theorems 2.3 and 3.1. The applications of our abstract results to semi-linear equations with the exponentially dichotomic linear parts are given in Section 4.

2. Bounded and periodic solutions to linear evolution equations

Given a function f taking values in a Banach space X having a separable predual Y (i.e., $X = Y'$ for a separable Banach space Y) we consider the non-homogeneous linear problem for the unknown function $u(t)$

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + f(t) \text{ for } t > 0 \\ u(0) = u_0 \in X, \end{cases} \tag{2.1}$$

where the family of partial differential operators $(A(t))_{t\geq 0}$ is given such that the homogeneous Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) \text{ for } t > s \geq 0 \\ u(s) = u_s \in X \end{cases} \tag{2.2}$$

is well-posed. By this we mean that there exists an evolution family $(U(t,s))_{t\geq s\geq 0}$ such that the solution of the Cauchy problem (2.2) is given by $u(t) = U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [10] (see also Nagel and Nickel [9] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$). We next give the precise concept of an evolution family in the following definition