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Fixed point properties for semigroups of nonlinear mappings on unbounded sets

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ABSTRACT

A well-known result of W. Ray asserts that if C is an unbounded convex subset of a Hilbert space, then there is a nonexpansive mapping $T: C \to C$ that has no fixed point. In this paper we establish some common fixed point properties for a semitopological semigroup S of nonexpansive mappings acting on a closed convex subset C of a Hilbert space, assuming that there is a point $c \in C$ with a bounded orbit and assuming that certain subspace of $C_b(S)$ has a left invariant mean. Left invariant mean (or amenability) is an important notion in harmonic analysis of semigroups and groups introduced by von Neumann in 1929 [28] and formalized by Day in 1957 [5]. In our investigation we use the notion of common attractive points introduced recently by S. Atsushiba and W. Takahashi.

1. Introduction

Let E be a Banach space and C be a nonempty bounded closed convex subset of E. The set C is said to have the fixed point property (abbreviated as fpp) if every nonexpansive mapping $T: C \to C$ has a fixed point, where T being nonexpansive means $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. The space E is said to have the fpp if every bounded closed convex set of E has the fpp.

A result of Browder [4] asserts that if a Banach space E is uniformly convex, then E has the fpp. As shown by Aspach [1] (see also [9, Example 11.2]), there is a weakly compact convex subset of $L^1[0,1]$ on which an isometry does not have a fixed point. It is also well-known that a weak* compact convex subset of $\ell^1(\mathbb{Z})$ has the fpp. However, $\ell^1(\mathbb{Z})$ does not have the fpp for bounded closed convex sets [9]. In a recent remarkable paper of Lin [25], it was shown that $\ell^1(\mathbb{Z})$ can be renormed to have the fpp. This answers in negative a long-standing open question of whether every Banach space with the fpp is necessarily reflexive.

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It was proved by B. Maurey in [26] that every nonempty weakly compact convex subset of the sequence space c_0 has the fpp for nonexpansive mappings. In the beautiful paper [7], T.D. Benavides proved that for every unbounded subset C in c_0 there is a nonexpansive mapping T on C which is fixed point free.

Let S be a semitopological semigroup, that is, a semigroup with a Hausdorff topology such that for each $t \in S$, the mappings $s \mapsto t \cdot s$ and $s \mapsto s \cdot t$ from S into S are continuous. Let C be a subset of a Banach space E. We say that $\mathcal{S} = \{T_s : s \in S\}$ is a *representation* of S on C if for each $s \in S$, T_s is a mapping from C into C and $T_{st}(x) = T_s(T_t x)$ $(s, t \in S,\ x \in C)$. Sometimes we simply use sx to denote $T_s(x)$ if there is no confusion in the context. The representation is called *separately* or, respectively, *jointly continuous* if the mapping $(s, x) \mapsto T_s(x)$ from $S \times C$ to C is separately or jointly continuous. We say that a representation $\mathcal{S}$ is *nonexpansive* if $\|T_s x - T_s y\| \le \|x - y\|$ for all $s \in S$ and all $x, y \in C$. A point $x \in C$ is called a *common fixed point* for (the representation of) S if $T_s(x) = x$ for all $s \in S$ (see [21] for more details). The set of all common fixed points for S in C is called the *fixed point set* of S (in C) and is denoted by $F(S)$.

Let $\mathcal{S}$ be a jointly continuous representation of S on a closed convex subset C of a Hilbert space H. Then, as is well-known, $F(S)$ is a closed and convex subset of C if it is not empty [6]. However, $F(S)$ may be empty for a continuous representation of S on an unbounded convex set C of a Hilbert space even if S is a commutative semigroup with a single generator [32].

In the recent paper [2], which was motivated by [33], Atsushiba and Takahashi introduced the concept of common attractive points for a nonexpansive representation $\mathcal{S}$ of a semigroup S on a set C in a Hilbert space H (precise definition may be seen in Section 2). They showed that $F(S) \ne \emptyset$ for commutative S if there is a common attractive point for S [2, Lemma 3.1]. They showed further that for commutative semigroups S, if $\{T_s c, s \in S\}$ is bounded for some $c \in C \subset H$, then the set $A_C(\mathcal{S})$ of all attractive points of $\mathcal{S}$ is not empty. As a consequence, $F(S) \ne \emptyset$ [2, Theorem 4.1]. We note that the assumption that $\{T_s c, s \in S\}$ is bounded for some $c \in C \subset H$ cannot be dropped even when S is commutative. Indeed, by the classical result of W. Ray in [32] as mentioned above, for every unbounded convex subset C of a Hilbert space there is a nonexpansive mapping T_0: $C \to C$ without a fixed point in C. In particular, the representation $\{T_n(c) : n \in \mathbb{N}\}$ of $(\mathbb{N}, +)$ does not have a common fixed point in C. An investigation continuing that of [2] may be seen in [34].

As one of the main results in this paper, we show that the result mentioned above of Atsushiba's and Takahashi remains true when $\mathcal{S}$ is a continuous representation of a left amenable semitopological semigroup S, where S being left amenable means that $C_b(S)$ of bounded continuous complex-valued functions on S has a left invariant mean. It also remains true when S is separable and left reversible if the representation is weakly equicontinuous. Here a semitopological semigroup S is *left reversible* if any two closed right ideals of S have non-void intersection, that is, $\overline{sS} \cap \overline{tS} \ne \emptyset$ for any $s, t \in S$, where, for a subset A of a topological space, $\overline{A}$ denotes the closure of A. This is the case when S is normal or $C_b(S)$ has a left invariant mean [15]. Likewise, S is *right reversible* if any two closed left ideals of S have non-void intersection, that is, $\overline{Ss} \cap \overline{St} \ne \emptyset$ for any $s, t \in S$. Left invariant mean (or amenability) is an important notion in harmonic analysis of semigroups and groups introduced by von Neumann in 1929 [28] and formalized by Day in 1957 [5].

The paper is organized as follows: In Section 3 we study the relation between the common attractive point and the common fixed point for a semigroup of nonexpansive mappings on a closed convex subset C of a strictly convex space. In Section 4 we establish our main results concerning common fixed points on a closed convex subset of a Hilbert space. In Section 5 we extend some of our results in Section 4 to the class of generalized hybrid mappings introduced recently in [17]. In Section 6 we post some related open problems.

2. Preliminaries and notations

Topologies considered in this paper will be all Hausdorff. Banach spaces are all assumed to be over the complex numbers $\mathbb{C}$. If E is a Banach space (resp. a dual Banach space), the weak topology (resp. weak* topology) of E will be denoted by wk (resp. wk*).