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Two porosity theorems for nonexpansive mappings in hyperbolic spaces

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ABSTRACT

In a previous paper of ours we used the notion of porosity to show that most of the nonexpansive self-mappings of bounded, closed and convex subsets of a Banach space are contractive and possess a unique fixed point which is the uniform limit of all iterates. In this paper we extend this result to nonexpansive self-mappings of closed and convex sets in a Banach space which are not necessarily bounded. As a matter of fact, it turns out that our results are true for all complete hyperbolic metric spaces.

1. Introduction and statement of main results

For more than fifty years now, there has been considerable interest in the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings in metric and Banach spaces. See, for instance, the papers and books by de Blasi and Myjak [2,3], Goebel and Kirk [4], Goebel and Reich [5], and Kirk [6], as well as the references mentioned therein. This interest originates in Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. Since that seminal result, many developments have taken place in this area. We mention, for example, existence results for fixed points of nonexpansive mappings which are not strictly contractive [4,5]. Such results were obtained for general nonexpansive mappings in special Banach space, while for self-mappings of general complete metric spaces existence results were established for, the so-called, contractive mappings [7]. For general nonexpansive mappings in general Banach spaces the existence of a unique fixed point was established in the generic sense, using the Baire category approach [2,3,11,14]. More precisely, in these works the space $\mathcal{A}$ of all nonexpansive self-mappings of a closed and convex set K in a Banach space is endowed with the natural metric of uniform convergence on bounded

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subsets, and it is shown that there exists a subset $\mathcal{A}' \subset \mathcal{A}$, which is a countable intersection of open and everywhere dense subsets of $\mathcal{A}$, such that every mapping in $\mathcal{A}'$ has a unique fixed point. Note that in [2,3] the set K was assumed to be bounded, while in [11] this assumption was removed.

As a matter of fact, it turns out that all these results are also true for nonexpansive self-mappings of closed and convex sets in complete hyperbolic spaces, an important class of metric spaces the definition of which we now recall.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding is called a *metric line*. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping is called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point will be denoted by $(1 - t)x \oplus ty$. We say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. This inequality, in its turn, implies that

$$\rho((1 - t)x \oplus ty, (1 - t)w \oplus tz) \le (1 - t)\rho(x, w) + t\rho(y, z)$$

for all x, y, z and w in X, and all $0 \le t \le 1$.

A set $K \subset X$ is called *ρ-convex* if $[x, y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in [5,8,9].

A property of elements of a complete metric space Z is said to be *generic* (typical) in Z if the set of all elements of Z which possess this property contains an everywhere dense G_δ subset of Z. In this case we also say that the property holds for a generic (typical) element of Z or that a generic (typical) element of Z has this property.

As we have already mentioned, a typical nonexpansive self-mapping of a bounded, closed and ρ-convex subset of a complete hyperbolic metric space has a unique fixed points which is the uniform limit of all its iterates. As a matter of fact, the subset of all those nonexpansive mappings which lack this property is small not only in the sense of Baire category, but also in the sense of porosity, a concept which we now recall.

Let Z be a complete metric space. We denote by $B_Z(y, r)$ the closed ball of center $y \in Z$ and radius $r > 0$. A subset $E \subset Z$ is called *porous* in Z if there exist numbers $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each number $r \in (0, r_0]$ and each point $y \in Z$, there exists a point $z \in Z$ for which

$$B_Z(z, \alpha r) \subset B_Z(y, r) \setminus E.$$

A subset of the space Z is called *σ-porous* in Z if it is a countable union of porous subsets of Z.