J. Math. Anal. Appl. 433 (2016) 1290–1304

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some properties of classes of real self-reciprocal polynomials

Vanessa Botta [a,*], Cleonice F. Bracciali [b], Junior A. Pereira [c]

[a] UNESP – Univ. Estadual Paulista, FCT, Departamento de Matemática e Computação, 19060-900, Presidente Prudente, SP, Brazil

[b] UNESP – Univ. Estadual Paulista, IBILCE, Departamento de Matemática Aplicada, 15054-000, São José do Rio Preto, SP, Brazil

[c] UNESP – Univ. Estadual Paulista, FCT, Programa de Pós-Graduação em Matemática Aplicada e Computacional, 19060-900, Presidente Prudente, SP, Brazil

ARTICLE INFO

Article history:
Received 14 May 2015
Available online 20 August 2015
Submitted by K. Driver

Keywords:
Self-reciprocal polynomials
Unit circle
Zeros
Monotonicity
Interlacing

ABSTRACT

The purpose of this paper is twofold. Firstly we investigate the distribution, simplicity and monotonicity of the zeros around the unit circle and real line of the real self-reciprocal polynomials $R_n^{(\lambda)}(z) = 1 + \lambda(z + z^2 + \cdots + z^{n-1}) + z^n$, $n \geq 2$ and $\lambda \in \mathbb{R}$. Secondly, as an application of the first results we give necessary and sufficient conditions to guarantee that all zeros of the self-reciprocal polynomials $S_n^{(\lambda)}(z) = \sum_{k=0}^{n} s_{n,k}^{(\lambda)} z^k$, $n \geq 2$, with $s_{n,0}^{(\lambda)} = s_{n,n}^{(\lambda)} = 1$, $s_{n,n-k}^{(\lambda)} = s_{n,k}^{(\lambda)} = 1 + k\lambda$, $k = 1, 2, \ldots, \lfloor n/2 \rfloor$ when n is odd, and $s_{n,n-k}^{(\lambda)} = s_{n,k}^{(\lambda)} = 1 + k\lambda$, $k = 1, 2, \ldots, n/2 - 1$, $s_{n,n/2}^{(\lambda)} = (n/2)\lambda$ when n is even, lie on the unit circle, solving then an open problem given by Kim and Park in 2008.

1. Introduction

Let the polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, $a_i \in \mathbb{C}$. Define the polynomial

$$P^*(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n = \bar{a}_0 \prod_{j=1}^{n} (z - z_j^*),$$

whose zeros z_k^* are the inverses of the zeros z_k of $P(z)$, that is, $z_k^* = 1/\bar{z}_k$.

If $P^*(z) = uP(z)$ with $|u| = 1$, then $P(z)$ is said to be a self-inversive polynomial, see [19]. If $P(z) = z^n P(1/z)$, then $P(z)$ is said to be self-reciprocal. If $a_i \in \mathbb{R}$, then $P(z)$ is called real self-reciprocal polynomial,

* Corresponding author.
E-mail addresses: botta@fct.unesp.br (V. Botta), cleonice@ibilce.unesp.br (C.F. Bracciali), junior.gusto@hotmail.com (J.A. Pereira).

http://dx.doi.org/10.1016/j.jmaa.2015.08.038
0022-247X/

see [14]. Notice that real self-reciprocal polynomials are also self-inversive polynomials. It is clear that if $P(z)$ is a self-reciprocal polynomial, then $a_i = a_{n-i}$, for $i = 0, 1, \ldots, n$.

The properties of self-reciprocal polynomials are interesting topics to study and have many applications in some areas of mathematics, see for example [9–11,13].

It is not difficult to verify that if a polynomial has all its zeros on the unit circle, then it is a self-inversive polynomial. The reciprocal is not always true, since self-inversive polynomials can have zeros that are symmetric with respect to the unit circle. The most famous result about the conditions for a self-inversive polynomial to have all its zeros on the unit circle is due to Conh, see [19, p. 18]: A necessary and sufficient condition for all the zeros of $P(z)$ to lie on the unit circle is that $P(z)$ is self-inversive and that all zeros of $P'(z)$ lie in or on this circle. In [4], Chen has given more flexible conditions than the Cohn's result. Choo and Kim in [7] gave an extension of Chen's result, to guarantee that the zeros on the unit circle are simple. Many authors have investigated special classes of self-inversive polynomials, see for example [14–17].

In [14], Kim and Park investigate the distribution of zeros around the unit circle of real self-reciprocal polynomials of even degree with five terms whose absolute values of middle coefficients equal the sum of all other coefficients. As a consequence of this study, they present a result related to the location of the zeros of the real self-reciprocal polynomial $S_n^{(\lambda)}(z) = \sum_{k=0}^{n} s_k^{(\lambda)} z^k$, with $s_k^{(\lambda)} = 1 + k\lambda$, for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$, for n odd and some values of $\lambda \in \mathbb{R}$ (see [14, Th. 7]). The authors remarked that for the three cases "$2 < \lambda < 2 + \frac{2}{\lfloor n/2 \rfloor}$ for $\lfloor n/2 \rfloor$ odd", "$\lambda = 2 + \frac{2}{\lfloor n/2 \rfloor}$ for $\lfloor n/2 \rfloor$ odd" and "$\lambda = -\frac{2}{\lfloor n/2 \rfloor}$" the location of the zeros of $S_n^{(\lambda)}(z)$ remains an open problem. Here, we give a complete proof about the location of the zeros of $S_n^{(\lambda)}(z)$ in the case n odd and $\lambda \in \mathbb{R}$, answering the open problems of [14, Th. 7] and we present a new result when n is even. Precisely, we will deal with the polynomials

$$S_n^{(\lambda)}(z) = \sum_{k=0}^{n} s_{n,k}^{(\lambda)} z^k, \quad n \geq 2, \tag{1}$$

with $s_{n,0}^{(\lambda)} = s_{n,n}^{(\lambda)} = 1$ and

$$\begin{aligned} &s_{n,k}^{(\lambda)} = s_{n,n-k}^{(\lambda)} = 1 + k\lambda,\ k = 1, 2, \ldots, \left\lfloor \tfrac{n}{2} \right\rfloor, \quad \text{if } n \text{ is odd,} \\ &s_{n,k}^{(\lambda)} = s_{n,n-k}^{(\lambda)} = 1 + k\lambda,\ k = 1, 2, \ldots, \tfrac{n}{2} - 1,\ s_{n,n/2}^{(\lambda)} = \tfrac{n}{2}\lambda, \quad \text{if } n \text{ is even.} \end{aligned} \tag{2}$$

The proofs of these results are obtained using properties of the polynomials

$$R_n^{(\lambda)}(z) = 1 + \lambda(z + z^2 + \cdots + z^{n-1}) + z^n, \quad n \geq 2, \tag{3}$$

with $\lambda \in \mathbb{R}$, studied in [1].

We denote the unit circle by $\mathcal{C} = \{z : z = e^{i\theta}, 0 \leq \theta \leq 2\pi\}$. For $z = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$, we consider the transformation

$$x = x(z) = \frac{z^{1/2} + z^{-1/2}}{2} = \cos(\theta/2). \tag{4}$$

In the context of orthogonal polynomials, see [6,12], the transformation (4) was first used by Delsarte and Genin in [8], and later, was further explored by Zhedanov in [23]. We also consider and present some properties of the zeros of the polynomials $W_n^{(\lambda)}(x)$ defined by

$$W_n^{(\lambda)}(x) = W_n^{(\lambda)}(x(z)) = z^{-n/2} R_n^{(\lambda)}(z), \quad \text{for } \ n \geq 1. \tag{5}$$