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Spectral analysis and synthesis on varieties ☆

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ABSTRACT

In this paper we prove that spectral synthesis holds for a variety on an Abelian group if spectral analysis holds on it and the residue ring of the annihilator of the variety is a Noether ring. This extends the fundamental result of M. Lefranc about spectral synthesis on finitely generated Abelian groups.

1. Introduction

In this note *ring* means a commutative ring with identity. If G is an Abelian group, then $\mathbb{C}G$ denotes the *group algebra* of G which is identified with the set of all finitely supported complex valued functions on G equipped with the pointwise linear operations and with the convolution. This is also identified with the dual of the locally convex topological vector space $\mathcal{C}(G)$ of all complex valued functions on G, equipped with the pointwise linear operations and with the topology of pointwise convergence. The space $\mathcal{C}(G)$ is considered as a $\mathbb{C}G$-module under the action of $\mathbb{C}G$ on $\mathcal{C}(G)$ defined by

$$\mu * f(x) = \sum_{y\in G} f(x-y)\mu(y)$$

whenever μ is in $\mathbb{C}G$, f is in $\mathcal{C}(G)$ and x is in G. The annihilator of a subset of $\mathcal{C}(G)$ in $\mathbb{C}G$ and the annihilator of a subset of $\mathbb{C}G$ in $\mathcal{C}(G)$ have their obvious meaning and they are denoted by $\operatorname{Ann} V$, and $\operatorname{Ann} I$, whenever $V \subseteq \mathcal{C}(G)$ and $I \subseteq \mathbb{C}G$, respectively. Closed submodules of $\mathcal{C}(G)$ are called *varieties* on G. For each variety V in $\mathcal{C}(G)$ and for each ideal I in $\mathbb{C}G$ we have

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$$\operatorname{Ann}(\operatorname{Ann} V) = V, \qquad \operatorname{Ann}(\operatorname{Ann} I) = I.$$

The intersection of all varieties containing the function $f : G \to \mathbb{C}$ is a variety, called the *variety generated by* f or simply the *variety of* f and denoted by $\tau(f)$.

Recall that a maximal ideal M of a ring R is an *exponential maximal ideal* if R/M is the complex field. The first theorem establishes some well-known facts on Noetherian group algebras.

Theorem 1. *Let G be an Abelian group. Then $\mathbb{C}G$ is Noetherian if and only if G is finitely generated. If G is finitely generated, then every maximal ideal of $\mathbb{C}G$ is exponential.*

Proof. Suppose that X is a finite generating set for G. Then $X \cup X^{-1}$ generates $\mathbb{C}G$, whence [1], Corollary 7.10 (p. 82) yields that the maximal ideals of $\mathbb{C}G$ are exponential.

For a subgroup H of G we regard $\mathbb{C}H$ as a subalgebra of $\mathbb{C}G$. Letting S be a set of coset representatives of H in G, the set I_H of finite sums of terms $\delta_s * \mu$ with s in S and μ in the augmentation ideal of $\mathbb{C}H$ is an ideal of $\mathbb{C}G$ that does not depend on the specific choice of S. If K is a proper subgroup of H, then I_K is properly contained in I_H. Hence, for $\mathbb{C}G$ to be Noetherian, G must be "subgroup Noetherian", i.e. every strictly ascending chain of subgroups must be finite. In particular, G is finitely generated. □

Spectral analysis for a variety means that each of its nonzero subvarieties includes an exponential. Spectral analysis on the group G means that spectral analysis holds for every variety on G. Spectral analysis holds for a nonzero variety if and only if every maximal ideal in $\mathbb{C}G$ containing the annihilator of the variety is exponential. In other words, spectral analysis holds for the nonzero variety V if and only if every maximal ideal of the ring $\mathbb{C}G/\operatorname{Ann} V$ is exponential. Hence, spectral analysis holds for G if and only if every maximal ideal of the group algebra $\mathbb{C}G$ is exponential. For these statements and for further references see [11,12].

We note that in 2005 M. Laczkovich and G. Székelyhidi characterized those discrete Abelian groups having spectral analysis (see [6]):

Theorem 2 *(M. Laczkovich, G. Székelyhidi). Spectral analysis holds on an Abelian group if and only if its torsion free rank is less than the continuum.*

Spectral synthesis studies the problem if finite dimensional subvarieties generate a given variety. We introduce a function class, the so-called exponential polynomials, which are exactly those generating finite dimensional varieties. Then the basic question on spectral synthesis is if a given variety is spanned by all exponential polynomials in it. Let G be an Abelian group. The function $\varphi : G \to \mathbb{C}$ is called an *exponential monomial*, if it can be represented in the following form:

$$\varphi(x) = P\big(a_1(x), a_2(x), \ldots, a_k(x)\big)m(x),$$

where $P : \mathbb{C}^n \to \mathbb{C}$ is a complex polynomial in k variables, $m : G \to \mathbb{C}$ is an exponential, and the functions $a_i : G \to \mathbb{C}$ are *additive functions*, that is, homomorphisms of G into the additive group of complex numbers. Linear combinations of exponential monomials are called *exponential polynomials*. Given a variety V on G we say that it is *synthesizable*, if all exponential monomials span a dense subspace in V. We say that *spectral synthesis* holds for V, if every subvariety of V is synthesizable. We say that spectral synthesis holds on G, if every variety on G is synthesizable. For these statements and for further references see [11,12]. The following theorem, due to M. Lefranc (see [8]), was the first general result about spectral synthesis on discrete Abelian groups.

Theorem 3 *(Lefranc). Spectral synthesis holds on $\mathbb{Z}^n$.*