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On optimal location of diffusion and related optimal control for null controllable heat equation ☆

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A R T I C L E I N F O

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We consider optimal locations of heat diffusion and related optimal control to achieve null controllability for multi-dimensional heat equations. Both time optimal control and norm optimal control problems are considered. The reason behind combining these two problems together is that these two problems are actually equivalent: The energy to be used to drive the system to zero in minimal time interval is actually the minimal energy of driving the system to zero in this minimal time interval and visa versa. We formulate the optimal locations for time optimal control and norm optimal control into two types of shape optimization problems. One is seeking the optimal domain of heat diffusion with a fixed interior actuator domain. This can be considered as a domain perturbation problem in shape optimization. Another is to seek both the optimal locations of the optimal heat diffusion domain and the related optimal actuator domain. The existences of these two types of shape optimization problems over some class of open sets in general $\mathbb{R}^N$ space have been proved separately.

1. Introduction

Compared with the lumped parameter control systems, a distinctive problem for systems described by the partial differential equations (PDEs) is a choice of the locations of the actuators [25]. Among them, the optimal location for the optimal controls that are used to optimize some performance of system is more significant. There are many different researches from different perspectives. Using a simple duct model, it is

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shown in [24] that the noise reduction performance depends strongly on actuator location. An approximation scheme is developed in [25] to find optimal locations of the optimal controls for abstract infinite-dimensional systems to minimize the cost functional with the worst choice of initial condition. A spillover phenomenon in optimal location of the actuators is presented in [16] for one-dimensional wave equation. The optimal shape and position of the actuators for stabilization of a string are discussed in [2,15]. From computational perspective, we refer to [10] and [30] and the references therein. Similar research is carried out under the name of moving sensors for which we refer to [4,8,20] and the references therein. However, there are very few examples on this aspect in literature for multi-dimensional PDEs [17]. In [26], a problem of optimizing the shape and position of the damping set of internal stabilization for a linear wave equation in $\mathbb{R}^N$, $N = 1, 2$ is considered. Very recently, a numerical approximation of null controls of the minimal L^1-norm for a linear heat equation with a bounded potential is considered in [27].

In this paper, we are concerned with two types of shape optimization problems associated with optimal control for multi-dimensional null controllable heat equation. Both time optimal control and norm optimal control problems are considered. The reason behind combining these two problems together is that these two problems are actually equivalent: The energy to be used to drive the system to zero in minimal time interval is actually the minimal energy of driving the system to zero in this minimal time interval and visa versa. This fact has been reviewed by many works, see, for instance, [32]. The existence of time optimal and norm optimal control problems for operational differential equations have been studied in [21], and elsewhere for time optimal control problems for heat equations in [28,31]. We formulate time optimal control or norm optimal control for heat equation into two types of the shape optimization problems. One is seeking the optimal domain of heat diffusion with a fixed interior actuator domain. This can be considered as a domain perturbation problem in shape optimization. The solution to this problem gives the optimal heat diffusion area with fixed internal control domain. Another is to seek both optimal domains of the actuator and the heat diffusion. The solution to this problem gives simultaneously the optimal locations of the actuator occupation and the heat diffusion area. The existences of associated shape optimization problems over some class of open sets in general $\mathbb{R}^N$ space have been proved separately, which is a central issue in shape optimization yet the uniqueness is secondary because more solutions give the engineers more options in design. With the regularity assumption for the boundary of unknown domain, it is reviewed in [5,11,12,29].

We denote by $B \subset \mathbb{R}^N$ a given open ball centered at origin with sufficiently larger radius, which contains all heat domains Ω that are discussed throughout the paper.

Suppose that $y_0 \in L^2(B)$ is a given initial value. Let us first recall the time optimal control problem for null controllable heat equation:

$$\begin{cases} \partial_t y(x,t) - \Delta y(x,t) = \chi_\omega(x) u(x,t) & \text{in } \Omega \times (0,+\infty), \\ y(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ y(x,0) = y_0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where ω is an open subset of Ω, $\chi_\omega(x)$ is the characteristic function of the domain ω, and $u \in \mathscr{U}_M$ is given by

$$\mathscr{U}_M = \{u : (0,+\infty) \to L^2(\Omega) \text{ is measurable and } \|u(\cdot,t)\|_{L^2(\Omega)} \le M \text{ for } t > 0 \text{ a.e.}\}$$

Denote by

$$T^M(\Omega,\omega) = \inf_{u \in \mathscr{U}_M} \{t;\ y(\cdot,t;u) = 0\} \tag{1.2}$$

the minimal time in the admissible control set $\mathscr{U}_M$ associated with ω and Ω, where $y(\cdot,t;u)$ denotes the solution of (1.1) associated with the control $u \in \mathscr{U}_M$. It is shown in [3,19] that for any given ω and Ω, there